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# ELEMENTS OF ANALYTIC GEOMETRY

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# ELEMENTS

OF

# ANALYTIC GEOMETRY

BY

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# PREFACE

As in most colleges the course in analytic geometry is preceded by a course in advanced algebra, it appeared desirable to publish separately those parts of our "Analytic Geometry and Principles of Algebra" which deal with analytic geometry, omitting the sections on algebra. This is done in the present work.

In plane analytic geometry, the idea of function is introduced as early as possible; and curves of the form $y = f(x)$, where $f(x)$ is a simple polynomial, are discussed even before the conic sections are treated systematically. This makes it possible to introduce the idea of the derivative; but the sections dealing with the derivative may be omitted.

In the chapters on the conic sections only the most essential properties of these curves are given in the text; thus, poles and polars are discussed only in connection with the circle.

The treatment of solid analytic geometry follows more the usual lines. But, in view of the application to mechanics, the idea of the vector is given some prominence; and the representation of a function of two variables by contour lines as well as by a surface in space is explained and illustrated by practical examples.

The exercises have been selected with great care in order not only to furnish sufficient material for practice in algebraic work, but also to stimulate independent thinking and to point out the applications of the theory to concrete problems. The number of exercises is sufficient to allow the instructor to make a choice.

ALEXANDER ZIWET.
LOUIS A. HOPKINS.

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# PLANE ANALYTIC GEOMETRY

## CHAPTER I

### COORDINATES

**1. Location of a Point on a Line.** The position of a point $P$ (Fig. 1) on a line is fully determined by its distance $OP$ from a fixed point $O$ on the line, if we know on which *side* of $O$ the point $P$ is situated (to the right or to the left of $O$ in Fig. 1). Let us agree, for instance, to count distances to the right of $O$ as positive, and distances to the left of $O$ as negative; this is indicated in Fig. 1 by the arrowhead which marks the *positive sense* of the line.

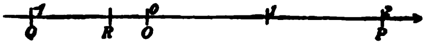

FIG. 1

The fixed point $O$ is called the *origin*. The distance $OP$, taken with the sign + if $P$ lies, let us say, on the right, and with the sign − when $P$ lies on the opposite side, is called the *abscissa* of $P$.

It is assumed that the unit in which the distances are measured (inches, feet, miles, etc.) is known. On a geographical map, or on a plan of a lot or building, this unit is indicated by the scale. In Fig. 1, the unit of measure is one inch, the abscissa of $P$ is $+2$, that of $Q$ is $-1$, that of $R$ is $-1/3$.

**2. Determination of a Point by its Abscissa.** Let us select, on a given line, an arbitrary *origin* $O$, a *unit of measure*, and a definite *sense* as positive. Then any real number, such as 5, $-3$, 7.35, $-\sqrt{2}$, regarded as the *abscissa* of a point $P$, fully determines the position of $P$ on the line. Conversely, every point on the line has one and only one abscissa.

The abscissa of a point is usually denoted by the letter $x$, which, in analytic geometry as in algebra, may represent any real or complex number.

To represent a *real* point the abscissa must be a real number. If in any problem the abscissa $x$ of a point is not a real number, there exists no real point satisfying the conditions of the problem.

### EXERCISES

**1.** What is the abscissa of the origin?

**2.** With the inch as unit of length, mark on a line the points whose abscissas are: 3, $-2$, $\sqrt{3}$, $-1.25$, $-\sqrt{5}$, $\frac{3}{4}$, $-\frac{1}{8}$.

**3.** On a railroad line running east and west, if the station B is 20 miles east of the station A and the station C is 33 miles east of A, what are the abscissas of A and C for B as origin, the sense eastward being taken as positive?

**4.** On a Fahrenheit thermometer, what is the positive sense? What is the unit of measure? What is the meaning of the reading 65°? What is meant by $-7°$?

**5.** A water gauge is a vertical post carrying a scale; the *mean* water level is generally taken as origin. If the water stands at $+7$ on one day and at $-11$ the next day, the unit being the inch, how much has the water fallen?

**6.** If $x_1$, $x_2$ (read: $x$ one, $x$ two) are the abscissas of any two points $P_1$, $P_2$ on a given line, show that the abscissa of the midpoint between $P_1$ and $P_2$ is $\frac{1}{2}(x_1 + x_2)$. Consider separately the cases when $P_1$, $P_2$ lie on the same side of the origin $O$ and when they lie on opposite sides.

**3. Ratio of Division.** A segment $AB$ (Fig. 2) of a straight line being given, it is shown in elementary geometry how to find the point $C$ that divides $AB$ in a given ratio $k$. Thus, if $k = \frac{2}{5}$, the point $C$ such that

$$\frac{AC}{AB} = \frac{2}{5}$$

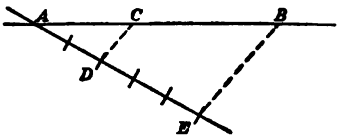

FIG. 2

is found as follows. On any line through $A$ lay off $AD = 2$ and $AE = 5$; join $B$ and $E$. Then the parallel to $BE$ through $D$ meets $AB$ at the required point $C$.

Analytically, the problem of dividing a line in a given ratio is solved as follows. On the line $AB$ (Fig. 3) we choose a point $O$ as origin and assign a positive sense. Then the abscissas $x_1$ of $A$ and $x_2$ of $B$ are known. To find a point $C$ which divides $AB$ in the *ratio of division* $k = AC/AB$, let us denote the unknown abscissa of $C$ by $x$. Then we have

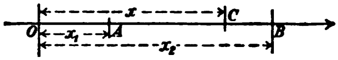

FIG. 3

$$AC = x - x_1, \quad AB = x_2 - x_1;$$

hence the abscissa $x$ of $C$ must satisfy the condition

$$\frac{x - x_1}{x_2 - x_1} = k,$$

whence

$$x = x_1 + k(x_2 - x_1);$$

or, if we write $\Delta x$ (read: delta $x$) for the "difference of the $x$'s," *i.e.* $\Delta x = x_2 - x_1$,

$$x = x_1 + k \cdot \Delta x.$$

Thus, if the abscissas of $A$ and $B$ are 2 and 7, the abscissas of the points that divide $AB$ in the ratios $\frac{1}{5}$, $\frac{1}{2}$, $\frac{6}{5}$, $\frac{3}{2}$ are 3, $4\frac{1}{2}$, 8, $9\frac{1}{2}$, respectively. Check these results by geometric construction.

If the segments $AC$ and $AB$ have the same sense, the division ratio $k$ is positive. For example, in Fig. 3, the point $C$ lies between $A$ and $B$; hence the division ratio $k$ is a positive proper fraction. If the division ratio $k$ is negative, the segments $AC$ and $AB$ must have opposite sense, so that $B$ and $C$ lie on the opposite sides of $A$.

If the abscissas of $A$ and $B$ are again 2 and 7, the abscissa $x$ of $C$ when $k = 2, -1, -\frac{2}{5}, -.2$ will be 12, $-3$, 0, 1, respectively. Illustrate this by a figure, and check by the geometric construction.

**4. Location of a Point in a Plane.** To locate a point in a plane, that is, to determine its position in a plane, we may proceed as follows. Draw two lines at right angles in the plane; on each of these take the point of intersection $O$ as origin, and assign a definite positive sense to each line, *e.g.* by marking each line with an arrowhead. It is usual to mark the positive sense of one line by affixing the letter $x$ to it, and the positive sense of the other line by affixing the letter $y$ to it, as in Fig. 4. These two lines are then called the ***axes of coordinates***, or simply the ***axes***. We distinguish them by calling the line $Ox$ the $x$-axis, or axis of abscissas, and the line $Oy$ the $y$-axis, or axis of ordinates. Now project the point $P$ on each axis, *i.e.* let fall the perpendiculars $PQ$, $PR$ from $P$ on the axes. The point $Q$ has the abscissa $OQ = x$ on the axis $Ox$. The point $R$ has the abscissa $OR = y$ on the axis $Oy$. The distance $OQ = RP = x$ is called the ***abscissa*** of $P$, and

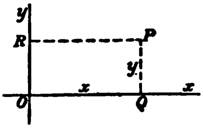

FIG. 4

$OR = QP = y$ is called the ***ordinate*** of $P$. The position of the point $P$ in the plane is fully determined if its abscissa $x$ and its ordinate $y$ are both given. The two numbers $x$, $y$ are also called the ***coordinates*** of the point $P$.

**5. Signs of the Coordinates. Quadrants.** It is clear from Fig. 4 that $x$ and $y$ are the perpendicular distances of the point $P$ from the two axes. It should be observed that each of these numbers may be positive or negative, as in § 1.

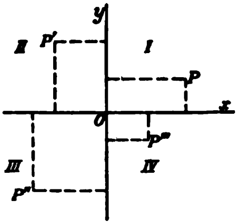

FIG. 5

The two axes divide the plane into four compartments distinguished as in trigonometry as the first, second, third, and fourth ***quadrants*** (Fig. 5). It is readily seen that any point in the first quadrant has both its coordinates positive. What are the signs of the coordinates in the other quadrants? What are the coordinates of the origin $O$? What are the coordinates of a point on one of the axes? It is customary to name the abscissa first and then the ordinate; thus the point $(-3, 5)$ means the point whose abscissa is $-3$ and whose ordinate is $5$.

*Every point in the plane has two definite real numbers as coordinates;* conversely, *to every pair of real numbers corresponds one and only one point of the plane.*

Locate the points: $(6, -2)$, $(0, 7)$, $(2-\sqrt{3}, \frac{2}{3})$, $(-4, 2\sqrt{2})$, $(-5, 0)$.

**6. Units.** It may sometimes be convenient to choose the unit of measure for the abscissa of a point different from the unit of measure for the ordinate. Thus, if the same unit, say one inch, were taken for abscissa and ordinate, the point $(3, 48)$ might fall beyond the limits of the paper. To avoid this we may lay off the ordinate on a scale of $\frac{1}{8}$ inch. When different units are used, the unit used on each axis should always be indicated in the drawing. When nothing is said to the contrary, the units for abscissas and ordinates are always understood to be the same.

**7. Oblique Axes.** The position of a point in a plane can also be determined with reference to two axes that are *not* at right angles; but the angle $\omega$ between these axes must be given (Fig. 6). The abscissa and the ordinate of the point $P$ are then the segments $OQ = x$, $OR = y$ cut off on the axes by the parallels through $P$ to the axes. If $\omega = \frac{1}{2}\pi$, *i.e.* if the axes are at right angles, we have the case of ***rectangular coordinates*** discussed in §§ 4, 5. In what follows, the axes are always taken at right angles unless the contrary is definitely stated.

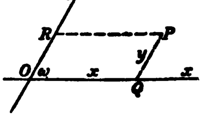

FIG. 6

**8. Distance of a Point from the Origin.** For the distance $r = OP$ (Fig. 7) of the point $P$ from the origin $O$ we have from the right-angled triangle $OQP$:

$$r = \sqrt{x^2 + y^2},$$

where $x$, $y$ are the coordinates of $P$.

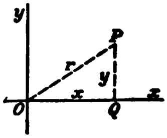

FIG. 7

If the axes are oblique (Fig. 8), with the angle $xOy = \omega$, we have, from the triangle $OQP$, in which the angle at $Q$ is equal to $\pi - \omega$,* by the cosine law of trigonometry,

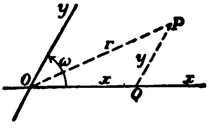

FIG. 8

$$r = \sqrt{x^2 + y^2 - 2\,xy \cos(\pi - \omega)} = \sqrt{x^2 + y^2 + 2\,xy \cos \omega}.$$

---

* In advanced mathematics, angles are generally measured in radians, the symbol $\pi$ denoting an angle of 180°.

Notice that these formulas hold not only when the point $P$ lies in the first quadrant, but quite generally wherever the point $P$ may be situated. Draw the figures for several cases.

**9. Distance between Two Points.** By Fig. 9, the distance $d = P_1P_2$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found if the coordinates of the two points are given. For in the triangle $P_1QP_2$ we have

$$P_1Q = x_2 - x_1, \quad QP_2 = y_2 - y_1;$$

hence

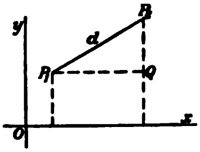

FIG. 9

$$(1) \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If we write $\Delta x$ (§ 3) for the "difference of the $x$'s" and $\Delta y$ for the "difference of the $y$'s", *i.e.*

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1,$$

the formula for the distance has the simple form

$$(2) \qquad d = \sqrt{(\Delta x)^2 + (\Delta y)^2};$$

or, in words,

*The distance between any two points is equal to the square root of the sum of the squares of the differences between their corresponding coordinates.*

Draw the figure showing the distance between two points (like Fig. 9) for various positions of these points and show that the expression for $d$ holds in all cases.

Show that the distance between two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ when the axes are oblique, with angle $\omega$, is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}$$
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2 + 2\,\Delta x \cdot \Delta y \cdot \cos\omega}.$$

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.. **Midpoint of a Segment.** *The midpoint $P$ of a segment* $_2$ *has for its coordinates the arithmetic means of the corre-* *ding coordinates of $P_1$ and $P_2$*; that is, if $x_1$, $y_1$ are the co- nates of $P_1$, $x_2$, $y_2$ those of $P_2$, the division ratio being $\frac{1}{2}$, the coordinates of the midpoint $P$ are (§ 10)

$$x = x_1 + \tfrac{1}{2}(x_2 - x_1) = \tfrac{1}{2}(x_1 + x_2),$$
$$y = y_1 + \tfrac{1}{2}(y_2 - y_1) = \tfrac{1}{2}(y_1 + y_2).$$

### EXERCISES

1. With reference to the same set of axes, locate the points (6, 4), $-\frac{1}{2}$), ($-$6.4, $-$3.2), ($-$4, 0), ($-$1, 5), (.001, $-$4.01).

2. Locate the points ($-$3, 4), (0, $-$1), (6, $-\sqrt{2}$), ($\frac{2}{3}$, $-10\frac{1}{2}$), $a$), ($a$, $b$), (3, $-$2), ($-$2, $\sqrt{2}$).

3. If $a$ and $b$ are positive numbers, in what quadrants do the follow g points lie: ($a$, $-b$), ($b$, $a$), ($a$, $a$), ($-b$, $b$), ($-b$, $-a$)?

4. Show that the points ($a$, $b$) and ($a$, $-b$) are symmetric with spect to the axis $Ox$; that ($a$, $b$) and ($-a$, $b$) are symmetric with re- ect to the axis $Oy$; that ($a$, $b$) and ($-a$, $-b$) are symmetric with spect to the origin.

5. In the city of Washington the lettered streets (A street, B street, tc.) run east and west, the numbered streets (1st street, 2d street, etc.) orth and south, the Capitol being the origin of coordinates. The axes f coördinates are called avenues, thus, *e.g.*, 1st street runs one block ast of the Capitol. If the length of a block were 1/10 mile, what would e the distance from the corner of South C street and East 5th street to the corner of North Q street and West 14th street?

6. Prove that the points (6, 2), (0, $-$6), (7, 1) lie on a circle whose center is (3, $-$2).

7. A square of side $s$ has its center at the origin and diagonals coincident with the axes; what are the coordinates of the vertices? of the midpoints of the sides?

8. If a point moves parallel to the axis $Oy$, which of its coordinates remains constant?

**9.** In what quadrants can a point lie if its abscissa is negative? its ordinate positive?

**10.** Find the coordinates of the points which trisect the distance between the points (1, – 2) and (– 3, 4).

**11.** To what point must the line segment drawn from (2, –3) to (–3, 5) be extended so that its length is doubled? trebled?

**12.** The abscissa of a point is – 3, its distance from the origin is 5; what is its ordinate?

**13.** A rectangular house is to be built on a corner lot, the front, 30 ft. wide, cutting off equal segments on the adjoining streets. If the house is 20 ft. deep, find the coordinates (with respect to the adjoining streets) of the back corners of the house.

**14.** A baseball diamond is 90 ft. square and pitcher's plate is 60 ft. from home plate. Using the foul lines as axes, find the coordinates of the following positions:

(*a*) pitcher's plate;

(*b*) catcher 8 ft. back of home plate and in line with second base;

(*c*) base runner playing 12 ft. from first base;

(*d*) third baseman playing midway between pitcher's plate and third base (before a bunt);

(*e*) right fielder playing 90 ft. from first and second base each.

**15.** How far does the ball go in Ex. 14 if thrown by third baseman in position (*d*) to second base?

**16.** If right fielder (Ex. 14) catches a ball in position (*e*) and throws it to third base for a double play, how far does the ball go?

**17.** A park 600 ft. long and 400 ft. wide has six lights arranged in a circle about a central light cluster. All the lights are 200 ft. apart, and the central cluster and two others are in a line parallel to the length of the park. What are the coordinates of all the lights with respect to two boundary hedges?

**18.** With respect to adjoining walks, three trees have coordinates (30 ft., 8 ft.), (20 ft., 45 ft.), (– 27 ft., 14 ft.), respectively. A tree is to be planted to form the fourth vertex of a parallelogram; where should it be placed? (Three possible positions; best found by division ratio.)

### 12. Area of a Triangle with One Vertex at the Origin.

Let one vertex of a triangle be the origin, and let the other vertices be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Draw through $P_1$ and $P_2$ lines parallel to the axes (Fig. 11). The area $A$ of the triangle is then obtained by subtracting from the area of the circumscribed rectangle the areas of the three non-shaded triangles; *i.e.*

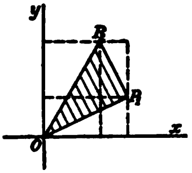

FIG. 11

$$A = x_1y_2 - \tfrac{1}{2}x_1y_1 - \tfrac{1}{2}x_2y_2 - \tfrac{1}{2}(x_1 - x_2)(y_2 - y_1)$$
$$= \tfrac{1}{2}(x_1y_2 - x_2y_1);$$

or, in determinant notation,

$$A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

This formula gives the area with the sign + or − according as the sense of the motion around the perimeter $OP_1P_2O$ is ***counterclockwise*** (opposite to the rotation of the hands of a clock) or ***clockwise***.

### 13. Translation of Axes.

Instead of the origin $O$ and the axes $Ox$, $Oy$ (Fig. 12), let us select a new origin $O'$ (read: $O$ prime) and new axes $O'x'$, $O'y'$, parallel to the old axes. Then any point $P$ whose coordinates with reference to the old axes are $OQ = x$, $QP = y$ will have with reference to the new axes the coordinates $O'Q' = x'$, $Q'P = y'$; and the figure shows that if $h$, $k$ are the coordinates of the new origin, then

$$x = x' + h,$$
$$y = y' + k.$$

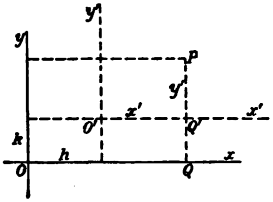

FIG. 12

The change from one set of axes to a new set is called a *transformation of coordinates*. In the present case, where the new axes are parallel to the old, this transformation can be said to consist in a *translation of the axes.*

**14. Area of Any Triangle.** Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of the triangle (Fig. 13). If we take one of these vertices, say $P_3$, as new origin, with the new axes parallel to the old, the new coordinates of $P_1, P_2$ will be:

$$x'_1 = x_1 - x_3,\quad x'_2 = x_2 - x_3,$$
$$y'_1 = y_1 - y_3,\quad y'_2 = y_2 - y_3.$$

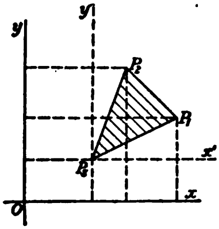

FIG. 13

Hence, by § 12, the area of the triangle $P_1P_2P_3$ is

$$A = \tfrac{1}{2}(x'_1y'_2 - x'_2y'_1) = \tfrac{1}{2}[(x_1-x_3)(y_2-y_3)-(x_2-x_3)(y_1-y_3)]$$
$$= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)];$$

or, in determinant notation,

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Here as in § 12 the sign of the area is + or − according as the sense of the motion along the perimeter $P_1P_2P_3P_1$ is counterclockwise.

### EXERCISES

**1.** Find the areas of the triangles having the following vertices:

(*a*) (1, 3), (5, 2), (4, 6); (*b*) (− 2, 1), (2, − 3), (0, − 6);
(*c*) (*a*, *b*), (*a*, 0), (0, *b*); (*d*) (4, 3), (6, − 2), (− 1, 5).

**2.** Show that the area of the triangle whose vertices are (7, − 8), (− 3, 2), (− 5, − 4) is four times the area of the triangle formed by joining the midpoints of the sides.

**3.** Find the area of the quadrilateral whose vertices are (2, 3), (− 1, − 1), (− 4, 2), (− 3, 6).

4. Find the area of the triangle whose vertices are $(a, 0)$, $(0, b)$, $(-c, -c)$.

5. Find the area of the triangle $(1, 4)$, $(3, -2)$, $(-3, 16)$. What does your result show about these points?

6. Find the area of the triangle $(a, b+c)$, $(b, c+a)$, $(c, a+b)$. What does the result show whatever the values of $a$, $b$, $c$?

7. Show that the points $(3, 7)$, $(7, 3)$, $(8, 8)$ are the vertices of an isosceles triangle. What is its area? Show that the same is true for the points $(a, b)$, $(b, a)$, $(c, c)$, whatever $a$, $b$, $c$, and find the area.

8. Find the perimeter of the triangle whose vertices are $(3, 7)$, $(2, -1)$, $(5, 3)$. Is the triangle scalene? What is its area?

9. Show that the area of a quadrilateral whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$ may be written in the form

$$A = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}.$$

**15. Statistics. Related Quantities.** If pairs of values of two related quantities are given, each of these pairs of values is represented by a point in the plane if the value of one quantity is represented by the abscissa and that of the other by the ordinate of the point. A curved line joining these points gives a vivid idea of the way in which the two quantities change. Statistics and the results of scientific experiments are often represented in this manner.

### EXERCISES

1. The population of the United States, as shown by the census reports, is approximately as given in the following table:

| YEAR | | 1800 | '10 | '20 | '30 | '40 | '50 | '60 | '70 | '80 | '90 | 1900 | '10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Million | 4 | 5 | 7 | 10 | 13 | 17 | 23 | 31 | 39 | 50 | 63 | 76 | 92 |

Mark the points corresponding to the pairs of numbers (1790, 4), (1800, 5), etc., on squared paper, representing the time on the horizontal axis and the population vertically. Connect these points by a curved line.

2. From the figure of Ex. 1, estimate approximately the population of the United States in 1875; in 1905; in 1915.

3. From the figure of Ex. 1, estimate approximately when the population was 25 millions; 60 millions; when it will be 100 millions.

4. Draw a figure to represent the growth of the population of your own State, from the figures given by the Census Reports.

[Other data suitable for statistical graphs can be found in large quantity in the Census Reports; in the Crop Reports of the government; in the quotations of the market prices of food and of stocks and bonds; in the World Almanac; and in many other books.]

5. The temperatures on a certain day varied hour by hour as follows:

| | A.M. | | | | | | N. | P.M. | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Time . . | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Temp. . . | 50 | 52 | 55 | 60 | 64 | 67 | 70 | 72 | 74 | 75 | 74 | 72 | 69 | 65 | 60 | 57 |

Draw a figure to represent these pairs of values.

6. In experiments on stretching an iron bar, the tension $t$ (in tons) and the elongation $E$ (in thousandths of an inch) were found to be as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ (in tons) . . . . . . . . . | 1 | 2 | 4 | 6 | 8 | 10 |
| $E$ (in thousandths of an inch) . | 10 | 19 | 38 | 60 | 81 | 103 |

Draw a figure to represent these pairs of values.

[Other data can be found in books on Physics and Engineering.]

7. By Hooke's law, the elongation $E$ of a stretched rod is supposed to be connected with the tension $t$ by the formula $E = c \cdot t$, where $c$ is a constant. Show that if $c = 10$, with the units of Ex. 6, the values of $E$ and $t$ would be nearly the same as those of Ex. 6. Plot the values given by the formula and compare with the figure of Ex. 6.

8. The distances through which a body will fall from rest in a vacuum in a time $t$ are given by the formula $s = 16\,t^2$, approximately, if $t$ is in seconds and $s$ is in feet. Show that corresponding values of $s$ and $t$ are

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ . . . . . . . | 1 | 2 | 3 | 4 | 5 | 6 |
| $s$ . . . . . . . | 16 | 64 | 144 | 256 | 400 | 576 |

Draw a figure to represent these pairs of values.

**16. Polar Coordinates.** The position of a point $P$ in a plane (Fig. 14) can also be assigned by its distance $OP = r$ from a fixed point, or ***pole***, $O$, and the angle $xOP = \phi$, made by the line $OP$ with a fixed line $Ox$, the ***polar axis***. The distance $r$ is called the ***radius vector***, the angle $\phi$ the ***polar angle*** (or also the *vectorial angle*, *azimuth*, *amplitude*, or *anomaly*) of the point $P$. The radius vector $r$ and the polar angle $\phi$ are called the ***polar coordinates*** of $P$.

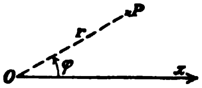

FIG. 14

Locate the points: $(5, \frac{1}{3}\pi)$, $(6, \frac{5}{3}\pi)$, $(2, 140°)$, $(7, 307°)$, $(\sqrt{5}, \pi)$, $(4, 0°)$.

To obtain for every point in the plane a single definite pair of polar coordinates it is ***sufficient*** to take the radius vector $r$ always positive and to regard as polar angle the positive angle between 0 and $2\pi$ ($0 \leq \phi < 2\pi$) through which the polar axis (regarded as a half-line or ray issuing from the pole $O$) must be turned about the pole $O$ in the counterclockwise sense to pass through $P$. The only exception is the pole $O$ for which $r = 0$, while the polar angle is indeterminate.

But it is not ***necessary*** to confine the radius vector to positive values and the polar angle to values between 0 and $2\pi$. A single definite point $P$ will correspond to every pair of real values of $r$ and $\phi$, if we agree that a negative value of the radius vector means that the distance $r$ is to be laid off in the negative sense on the polar axis, after being turned through the angle $\phi$, and that a negative value of $\phi$ means that the polar axis should be turned in the clockwise sense.

The polar angle is then not changed by adding to it any positive or

**negative integral multiple of $2\pi$; and a point whose polar coordinates are $r$, $\phi$ can also be described as having the coordinates $-r$, $\phi \pm \pi$.**

Locate the points:

$$(3, -\tfrac{1}{2}\pi),\ (a, -\tfrac{2}{3}\pi),\ (-5, 75°),\ (-3, -20°).$$

**17. Transformation from Cartesian to Polar Coordinates,** and *vice versa.* The coordinates $OQ = x$, $QP = y$, defined in § 4, are called ***cartesian*** coordinates, to distinguish them from the polar coordinates. The term is derived from the Latin form, Cartesius, of the name of RENÉ DESCARTES, who first applied the method of coordinates systematically (1637), and thus became the founder of analytic geometry.

The relation between the cartesian and polar coordinates of one and the same point $P$ appears from Fig. 15. We have evidently:

$$\begin{cases} x = r\cos\phi, \\ y = r\sin\phi, \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan\phi = \dfrac{y}{x}. \end{cases}$$

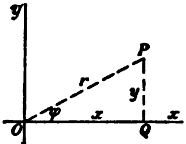

FIG. 15

**18. Distance between Two Points in Polar Coordinates.** If two points $P_1$, $P_2$ are given by their polar coordinates, $r_1$, $\phi_1$ and $r_2$, $\phi_2$, the distance $d = P_1P_2$ between them is found from the triangle $OP_1P_2$ (Fig. 16), by the cosine law of trigonometry, if we observe that the angle at $O$ is equal to $\pm(\phi_2 - \phi_1)$:

$$d = \sqrt{r_1^2 + r_2^2 - 2\,r_1r_2\cos(\phi_2 - \phi_1)}.$$

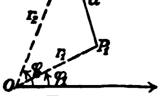

FIG. 16

### EXERCISES

**1.** Find the distances between the points: $(2, \tfrac{1}{6}\pi)$ and $(4, \tfrac{3}{4}\pi)$; $(a, \tfrac{1}{2}\pi)$ and $(3a, \tfrac{1}{6}\pi)$.

**2.** Find the cartesian coordinates of the points $(5, \tfrac{1}{4}\pi)$, $(6, -\tfrac{1}{4}\pi)$, $(4, \tfrac{1}{3}\pi)$, $(2, \tfrac{2}{3}\pi)$, $(7, \pi)$, $(6, -\pi)$, $(4, 0)$, $(-3, 60°)$, $(-5, -90°)$.

3. Find the polar coordinates of the points $(\sqrt{3}, 1)$, $(-\sqrt{3}, 1)$, $(1, -1)$, $(-\frac{1}{2}, -\frac{1}{2})$, $(-a, a)$.

4. Find an expression for the area of a triangle whose vertices are $(0, 0)$, $(r_1, \phi_1)$, and $(r_2, \phi_2)$.

5. Find the area of the triangle whose vertices are $(r_1, \phi_1)$, $(r_2, \phi_2)$, $(r_3, \phi_3)$.

**19. Projection of Vectors.** A straight line segment $AB$ of definite length, direction, and *sense* (indicated by an arrowhead, pointing from $A$ to $B$) is called a *vector*. The *projection* $A'B'$ (Figs. 17, 18) *of a vector* $AB$ *on an axis, i.e.* on a line $l$ on which a definite sense has been selected as positive, *is the product of the length of the vector* $AB$ *into the cosine of the angle between the positive senses of the axis and the vector:*

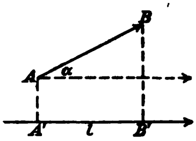

FIG. 17

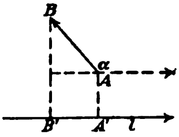

FIG. 18

$$A'B' = AB \cos \alpha.$$

The positive sense of the axis makes with the vector two angles whose sum is $2\pi = 360°$. As their cosines are the same, it makes no difference which of the two angles is used.

With these conventions it is readily seen that *the sum of the projections of the sides of an open polygon on any axis is equal to the projection of the closing side on the same axis*, the sides of the open polygon being taken in the same sense around the perimeter. Thus, in Fig. 19, the vectors $P_1P_2$, $P_2P_3$, $\cdots$ $P_5P_6$ are in-

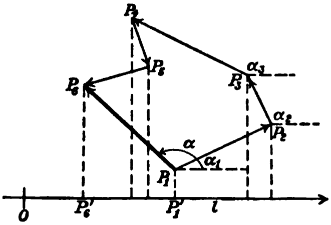

FIG. 19

clined at the angles $\alpha_1, \alpha_2, \cdots \alpha_5$ to the axis $l$; the closing line $P_1P_6$ makes the angle $\alpha$ with $l$; its projection is $P'_1P'_6$; and we have

$$P_1P_2 \cos \alpha_1 + P_2P_3 \cos \alpha_2 + P_3P_4 \cos \alpha_3 + P_4P_5 \cos \alpha_4 + P_5P_6 \cos \alpha_5 \\ = P'_1P'_6 = P_1P_6 \cos \alpha.$$

For, if the abscissas of $P_1, P_2, \cdots P_6$ measured along $l$, from any origin $O$ on $l$, are $x_1, x_2, \cdots x_6$, the projections of the vectors are $x_2 - x_1$, $x_3 - x_2$, etc., so that our equation becomes the identity:

$$x_2 - x_1 + x_3 - x_2 + x_4 - x_3 + x_5 - x_4 + x_6 - x_5 = x_6 - x_1.$$

**20. Components and Resultants of Vectors.** In physics, forces, velocities, accelerations, etc., are represented by vectors because such magnitudes have not only a numerical value but also a definite direction and sense.

According to the *parallelogram law* of physics, two forces $OP_1$, $OP_2$, acting on the same particle, are together equivalent to the single force $OP$ (Fig. 20), whose vector is the diagonal of the parallelogram formed with $OP_1$, $OP_2$ as adjacent sides. The same law holds for simultaneous velocities and accelerations, and for simultaneous or consecutive rectilinear translations. The vector $OP$ is called the ***resultant*** of $OP_1$ and $OP_2$, and the vectors $OP_1$, $OP_2$ are called the ***components*** of $OP$.

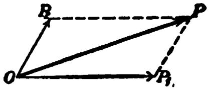

FIG. 20

To construct the resultant it suffices to lay off from the extremity of the vector $OP_1$ the vector $P_1P = OP_2$; the closing line $OP$ is the resultant. This leads at once to finding the resultant $OP$ of any number of

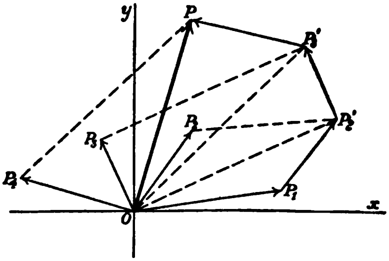

FIG. 21

vectors, by *adding* the component vectors *geometrically*, *i.e.* putting them together endwise successively, as in Fig. 21, where the dotted lines need not be drawn.

By § 19, the projection of the resultant on any axis is equal to the sum of the projections of all the components on the same axis.

### EXERCISES

**1.** The cartesian coordinates $x$, $y$ of any point $P$ are the projections of its radius vector $OP$ on the axes $Ox$, $Oy$. (See § 16.)

**2.** The projection of any vector $AB$ on the axis $Ox$ is the difference of the abscissas of $A$ and $B$; similarly for $Oy$.

**3.** A force of 10 lb. is inclined to the horizon at 60°; find its horizontal and vertical components.

**4.** A ship sails 40 miles N. 60° E. then 24 miles N. 45° E. How far is the ship then from its starting point? How far east? How far north?

**5.** A point moves 5 ft. along one side of an equilateral triangle, then 6 ft. parallel to the second, and finally 8 ft. parallel to the third side. What is the distance from the starting point?

**6.** The sum of the projections of the sides of any *closed* polygon on any axis is zero.

**7.** If three forces acting on a particle are parallel and proportional to the sides of a triangle, the forces are in equilibrium, *i.e.* their resultant is zero. Similarly for any closed polygon.

**8.** Find the resultant of the forces $OP_1$, $OP_2$, $OP_3$, $OP_4$, $OP_5$, if the coordinates of $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, with $O$ as origin, are $(3, 1)$, $(1, 2)$, $(-1, 3)$, $(-2, -2)$, $(2, -2)$. (Resolve each force into its components along the axes.)

**9.** If any number of vectors (in the same plane), applied at the origin, are given by the coordinates $x$, $y$ of their extremities, the length of the resultant is $=\sqrt{(\Sigma x)^2+(\Sigma y)^2}$ (where $\Sigma x$ means the sum of the abscissas, $\Sigma y$ the sum of the ordinates), and its direction makes with $Ox$ an angle $\alpha$ such that $\tan \alpha = \Sigma y / \Sigma x$.

**10.** Find the horizontal and vertical components of the velocity of a ball when moving 200 ft./sec. at an angle of 30° to the horizon.

**11.** Six forces of 1, 2, 3, 4, 5, 6 lb., making angles of 60° each with the next, are applied at the same point, in a plane; find their resultant.

**12.** A particle at one vertex of a square is acted upon by three forces represented by the vectors from the particle to the other three vertices; find the resultant.

**21. Geometric Propositions.** In using analytic geometry to prove general geometric propositions, it is generally convenient to select as origin a prominent point in the geometric figure, and as axes of coordinates prominent lines of the figure.

Example. In any right triangle, the distance from the vertex of the right angle to the midpoint of the hypotenuse is equal to half the hypotenuse.

Since this theorem is true, if at all, when the triangle is in any position, we may place the vertex of the right angle at the origin and the adjacent sides along the positive axes. Then the coordinates of the vertices are $(0, 0)$, $(a, 0)$, $(0, b)$, where $a$ and $b$ are the lengths of the two sides about the right angle.

The length of the hypotenuse is $\sqrt{a^2 + b^2}$. The midpoint of the hypotenuse has the coordinates $(a/2, b/2)$, by § 11. Hence the distance from this point to the vertex of the right angle $(0, 0)$ is $\sqrt{(a/2)^2+(b/2)^2} = \frac{1}{2}\sqrt{a^2+b^2}$. Since this is half the length of the hypotenuse, the theorem is proved.

Sometimes greater symmetry and elegance is gained by taking the coordinate system in a general position.

### MISCELLANEOUS EXERCISES

**1.** A regular hexagon of side 1 has its center at the origin and one diagonal coincident with the axis $Ox$; find the coordinates of the vertices.

**2.** If a square, with each side 5 units in length, is placed with one vertex at the origin and a diagonal coincident with the axis $Ox$, what are the coordinates of the vertices?

**3.** If a rectangle, with two sides 3 units in length and two sides $3\sqrt{3}$ units in length, is placed with one vertex at the origin and a diagonal along the axis $Ox$, what are the coordinates of the vertices? There are two possible positions of the rectangle; give the answers in both cases.

4. Show that the points $(0, -1)$, $(-2, 3)$, $(6, 7)$, $(8, 3)$ are the vertices of a parallelogram. Prove that this parallelogram is a *rectangle*.

5. Show that the points $(1, 1)$, $(-1, -1)$, $(+\sqrt{3}, -\sqrt{3})$ are the vertices of an equilateral triangle.

6. Show that the points $(6, 6)$, $(3/2, -3)$, $(-3, 12)$, $(-\frac{15}{2}, 3)$ are the vertices of a parallelogram.

7. Find the radius and the coordinates of the center of the circle passing through the three points $(2, 3)$, $(-2, 7)$, $(0, 0)$.

8. The vertices of a triangle are $(0, 6)$, $(4, -3)$, $(-5, 6)$. Find the lengths of the medians and the coordinates of the *centroid* of the triangle, *i.e.* of the intersection of the medians.

Prove the following propositions:

9. The diagonals of any rectangle are equal.

10. The distance between the midpoints of two sides of any triangle is equal to half the third side.

11. The distance between the midpoints of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.

12. The line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

13. If two medians of a triangle are equal, the triangle is isosceles.

14. In any triangle the sum of the squares of any two sides is equal to twice the square of the median drawn to the midpoint of the third side plus half the square of the third side.

15. The line segments joining the midpoints of the opposite sides of any quadrilateral bisect each other.

16. The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line segment joining the midpoints of the diagonals.

17. The difference of the squares of any two sides of a triangle is equal to the difference of the squares of their projections on the third side.

18. The vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ of a triangle being given, find the *centroid* (intersection of medians).

# CHAPTER II

## THE STRAIGHT LINE

**22. Line Parallel to an Axis.** When the coordinates $x$, $y$ of a point $P$ with reference to given axes $Ox$, $Oy$ are known, the position of $P$ in the plane of the axes is determined completely and uniquely. Suppose now that only one of the coordinates is given, say, $x = 3$; what can be said about the position of the point $P$? It evidently lies somewhere on the line $AB$ (Fig. 22) that is parallel to the axis $Oy$ and has the distance 3 from $Oy$. Every point of the line $AB$ has an abscissa $x = 3$, and every point whose abscissa is 3 lies on the line $AB$. For this reason we say that the equation

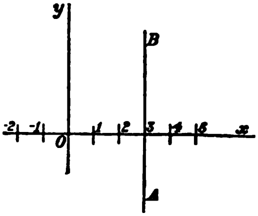

FIG. 22

$$x = 3$$

*represents the line AB;* we also say that $x = 3$ is *the equation of the line AB.*

More generally, the equation $x = a$, where $a$ is any real number, represents that parallel to the axis $Oy$ whose distance from $Oy$ is $a$. Similarly, the equation $y = b$ represents a parallel to the axis $Ox$.

### EXERCISES

Draw the lines represented by the equations:

1. $x = -2$.
2. $x = 0$.
3. $x = 12.5$.
4. $5x = 7$.
5. $y = 0$.
6. $2y = -7$.
7. $3x + \frac{1}{2} = 0$.
8. $10 - 3y = 0$.
9. $y = \pm 2$.

**23. Line through the Origin.** Let us next consider any line* through the origin $O$, such as the line $OP$ in Fig. 23. The points of this line have the property that the ratio $y/x$ of their coordinates is the same, wherever on this line the point $P$ be taken. This ratio is equal to the tangent of the angle $\alpha$ made by the line with the axis $Ox$, *i.e.* to what we shall call the ***slope*** of the line. Let us put

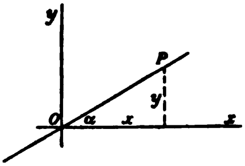

FIG. 23

$$\tan \alpha = m;$$

then we have, for any point $P$ on this line: $y/x = m$, *i.e.*:

$$y = mx. \tag{1}$$

Moreover, for any point $Q$, not on this line, the ratio $y/x$ must evidently be different from $\tan \alpha$, *i.e.* from $m$. The equation $y = mx$ is therefore said to *represent the line through O whose slope is m;* and $y = mx$ is called *the equation of this line.* We mean by this statement that the relation $y = mx$ is satisfied by the coordinates of every point on the line $OP$, and only by the coordinates of the points on this line. Notice in particular that the coordinates of the origin $O$, *i.e.* $x = 0$, $y = 0$, satisfy the equation $y = mx$.

**24. Proportional Quantities.** Any two values of $x$ are *proportional* to the corresponding values of $y$ if $y = mx$. For, if $(x_1, y_1)$ and $(x_2, y_2)$ are two pairs of values of $x$ and $y$ that satisfy (1), we have

$$y_1 = mx_1, \quad y_2 = mx_2;$$

hence, dividing,

$$y_1/y_2 = x_1/x_2.$$

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* For the sake of brevity, a *straight line* will here in general be spoken of simply as a *line*; a line that is not straight will be called a *curve*.

The constant quantity $m$ is called the *factor of proportionality.*

Many instances occur in mathematics and in the applied sciences of two quantities related to each other in this manner. It is often said that one quantity $y$ *varies as* the other quantity $x$.

Thus Hooke's Law states that the elongation $E$ of a stretched wire or spring varies as the tension $t$; that is, $E = kt$, where $k$ is a constant.

Again, the circumference $c$ of a circle varies as the radius $r$; that is,

$$c = 2\pi r.$$

## EXERCISES

**1.** Draw each of the lines :

(*a*) $y = 2x$. (*c*) $y = -\frac{7}{12}x$. (*e*) $5x + 3y = 0$. (*g*) $y = -x$.
(*b*) $y = -3x$. (*d*) $5y = 3x$. (*f*) $y = x$. (*h*) $x - y = 0$.

**2.** Show that the equation $ax + by = 0$ can be reduced to the form $y = mx$, if $b \neq 0$, and therefore represents a line through the origin.

**3.** Find the slope of the lines :

(*a*) $x + y = 0$. (*c*) $3x - \frac{1}{10}y = 0$.
(*b*) $x - y = 0$. (*d*) $\sqrt{2}x + y = 0$.

**4.** Draw a line to represent Hooke's Law $E = kt$, if $k = 10$ (see Ex. 7, p. 14). Let $t$ be represented as horizontal lengths (as is $x$ in § 23) and let $E$ be represented by vertical lengths (as is $y$ in § 23).

**5.** Draw a line to represent the relation $c = 2\pi r$, where $c$ means the circumference and $r$ the radius of a circle.

**6.** The number of yards $y$ in a given length varies as the number of feet $f$ in the same length; in particular, $f = 3y$. Draw a figure to represent this relation.

**7.** If 1 in. = 2.54 cm., show that $c = 2.54\,i$, where $c$ is the number of centimeters and $i$ is the number of inches in the same length. Draw a figure.

**25. Slope Form.** Finally, consider a line that does not pass through the origin and is not parallel to either of the axes of coordinates (Fig. 24); let it intersect the axes $Ox$, $Oy$ at $A$, $B$, respectively, and let $P\ (x, y)$ be any other point on it. The figure shows that *the slope $m$ of the line, i.e.* the tangent of the angle $\alpha$ at which the line is inclined to the axis $Ox$, is

$$m = \tan\alpha = \frac{RP}{BR};$$

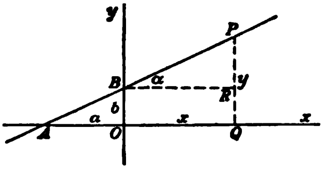

FIG. 24

or, since $RP = QP - QR = QP - OB = y - b$ and $BR = OQ = x$:

$$m = \frac{y-b}{x};$$

that is,

$$y = mx + b, \tag{2}$$

where $b = OB$ is called the ***intercept*** made by the line on the axis $Oy$, or briefly the ***y-intercept.***

The *slope angle* $\alpha$ at which the line is inclined to the axis $Ox$ is always understood as the smallest angle through which the positive half of the axis $Ox$ must be turned counterclockwise about the origin to become parallel to the line.

**26. Equation of a Line.** On the line $AB$ of Fig. 24 take any other point $P'$; let its coordinates be $x'$, $y'$, and show that

$$y' = mx' + b.$$

Take the point $P'\ (x', y')$ outside the line $AB$ and show that the equation $y = mx + b$ is *not* satisfied by the coordinates $x'$, $y'$ of such a point.

For these reasons the equation $y = mx + b$ is said to *represent the line whose y-intercept is $b$ and whose slope is $m$;* it is also called *the equation of this line.* The $y$-intercept $OB = b$ and the slope $m = \tan\alpha$ together fully determine the line.

*Every line of the plane can be represented by an equation of the form*

$$y = mx + b,$$

*excepting the lines parallel to the axis Oy.* When the line becomes parallel to the axis $Oy$, both its slope $m$ and its $y$-intercept $b$ become infinite. We have seen in § 22 that the equation of a line parallel to the axis $Oy$ is of the form $x = a$.

Reduce the equation $3x - 2y = 5$ to the form $y = mx + b$ and sketch the line.

## EXERCISES

**1.** Sketch the lines whose $y$-intercept is $b = 2$ and whose slopes are $m = \frac{1}{2}, 3, 0, -\frac{2}{3}$; write down their equations.

**2.** Sketch the lines whose slope is $m = 4/3$ and whose $y$-intercepts are 0, 1, 2, 5, − 1, − 2, − 6, − 12.2, and write down their equations.

**3.** Sketch the lines whose equations are:

(*a*) $y=2x+3.$ (*c*) $y=x-\frac{1}{2}.$ (*e*) $x-y=1.$ (*g*) $7x-y+12=0.$
(*b*) $y=-\frac{1}{2}x+1.$ (*d*) $x+y=1.$ (*f*) $x-2y+2=0.$ (*h*) $4x+3y+5=0.$

**4.** Do the points (1, 5), (−2, −1), (3, 7) lie on the line $y=2x+3$?

**5.** A cistern that already contained 300 gallons of water is filled at the rate of 100 gallons per hour. Show that the amount $A$ of water in the cistern $n$ hours after filling begins is $A = 100n + 300$. Draw a figure to represent this relation, plotting the values of $A$ vertically, with 1 vertical space = 100 gallons.

**6.** In experiments with a pulley block, the pull $p$ in lbs., required to lift a load $l$ in lbs., was found to be expressed by the equation $p = .15l + 2$. Draw this line. How much pull is required to operate the pulley with no load (*i.e.* when $l = 0$)?

**7.** The readings of a gas meter being tested, $T$, were found in comparison with those of a standard gas meter $S$, and the two readings satisfied the equation $T = 300 + 1.2S$. Draw a figure. What was the reading $T$ when the reading $S$ was zero? What is the meaning of the slope of the line in the figure?

**27. Parallel and Perpendicular Lines.** Two lines

$$y = m_1x + b_1, \quad y = m_2x + b_2$$

are obviously ***parallel*** if they have the same slope, *i.e.* if

$$m_1 = m_2. \tag{3}$$

Two lines $y = m_1x + b_1$, $y = m_2x + b_2$ are ***perpendicular*** if the slope of one is equal to minus the reciprocal of the slope of the other, *i.e.* if

$$m_1m_2 = -1. \tag{4}$$

For if $m_1 = \tan\alpha_1$, $m_2 = \tan\alpha_2$, the condition that $m_1m_2 = -1$ gives $\tan\alpha_2 = -1/\tan\alpha_1 = -\cot\alpha_1$, whence $\alpha_2 = \alpha_1 + \frac{1}{2}\pi$.

### EXERCISES

**1.** Write down the equation of any line: (*a*) parallel to $y = 3x - 2$, (*b*) perpendicular to $y = 3x - 2$.

**2.** Show that the parallel to $y = 3x - 2$ through the origin is $y = 3x$.

**3.** Show that the perpendicular to $y = 3x - 2$ through the origin is

$$y = -\tfrac{1}{3}x.$$

**4.** For what value of $b$ does the line $y = 3x + b$ pass through the point (4, 1)? Find the parallel to $y = 3x - 2$ through the point (4, 1).

**5.** Find the parallel to $y = 5x + 1$ through the point (2, 3).

**6.** Find the perpendicular to $y = 2x - 1$ through the point (1, 4).

**7.** What is the geometrical meaning of $b_1 = b_2$ in the equations

$$y = m_1x + b_1, \quad y = m_2x + b_2?$$

**8.** Two water meters are attached to the same water pipe and the water is allowed to flow steadily through the pipe. The readings $R_1$ and $R_2$ of the two meters are found to be connected with the time $t$ by means of the equations

$$R_1 = 2.5\,t, \quad R_2 = 2.5\,t + 150,$$

where $R_1$ and $R_2$ are measured in cubic feet and $t$ is measured in seconds. Show that the lines that represent these equations are parallel. What is the meaning of this fact?

**9.** The equations connecting the pull $p$ required to lift a load $w$ is found for two pulley blocks to be

$$p_1 = .05\,w + 2, \quad p_2 = .05\,w + 1.5$$

Show that the lines representing these equations are parallel. Explain.

10. The equations connecting the pull $p$ required to lift a load $w$ is found for two pulley blocks to be

$$p_1 = .15\,w + 1.5, \quad p_2 = .05\,w + 1.5.$$

Show that the lines representing these equations are not parallel, but that the values of $p_1$ and $p_2$ are equal when $w = 0$. Explain.

**28. Linear Function.** The equation $y = mx + b$, when $m$ and $b$ are given, assigns to every value of $x$ one and only one definite value of $y$. This is often expressed by saying that $mx + b$ is a ***function*** of $x$; and as the expression $mx + b$ is of the first degree in $x$, it is called a *function of the first degree* or, owing to its geometrical meaning, a ***linear function*** of $x$.

Examples of functions of $x$ that are *not linear* are $3x^2 - 5$, $ax^2 + bx + c$, $x(x-1)$, $1/x$, $\sin x$, $10^x$, etc. The equations $y = 3x^2 - 5$, $y = ax^2 + bx + c$, etc., represent, as we shall see later, not straight lines but curves.

The linear function $y = mx + b$, being the most simple kind of function, occurs very often in the applications. Notice that the constant $b$ is the value of the function for $x = 0$. The constant $m$ is the ***rate of change*** of $y$ with respect to $x$.

**29. Illustrations.** EXAMPLE 1. A man, on a certain date, has \$10 in bank; he deposits \$3 at the end of every week; how much has he in bank $x$ weeks after date?

Denoting by $y$ the number of dollars in bank, we have

$$y = 3x + 10.$$

His deposit at any time $x$ is a linear function of $x$. Notice that the coefficient of $x$ gives the ***rate of increase*** of this deposit; in the graph this is the ***slope*** of the line.

EXAMPLE 2. Water freezes at 0° C. and 32° F.; it boils at 100° C. and at 212° F.; assuming that mercury expands uniformly, *i.e.* proportionally to the temperature, and denoting by $x$ any temperature in Centigrade degrees, by $y$ the same temperature in Fahrenheit degrees, we have

$$\frac{y-32}{x}=\frac{212-32}{100}=\frac{9}{5}, \text{ i.e. } y=\tfrac{9}{5}x+32.$$

If the line represented by this equation be drawn accurately, on a sufficiently large scale, it could be used to convert centigrade temperature into Fahrenheit temperature, and *vice versa.*

EXAMPLE 3. A rubber band, 1 ft. long, is found to stretch 1 in. by a suspended mass of 1 lb. Let the suspended mass be increased by 1 oz., 2 oz., etc., and let the corresponding lengths of the band be measured. Plotting the masses as abscissas and the lengths of the band as ordinates, it will be found that the points $(x, y)$ lie very nearly on a straight line whose equation is $y=\frac{1}{12}x+1$. The experimental fact that the points lie on a straight line, *i.e.* that the function is linear, means that the *extension*, $y-1$, is proportional to the *tension*, *i.e.* to the weight of the suspended mass $x$ (Hooke's Law).

Notice that only the part of the line in the first quadrant, and indeed only a portion of this, has a physical meaning. Can this range be extended by using a spiral steel spring?

EXAMPLE 4. When a point $P$ moves along a line so as to describe always equal spaces in equal times, its motion is called *uniform.* The spaces passed over are then proportional to the times in which they are described, and the coefficient of proportionality, *i.e.* the ratio of the distance to the time, is called the *velocity* $v$ of the uniform motion. If at the time $t=0$ the moving point is at the distance $s_0$, and at the time $t$ at the distance $s$, from the origin, then

$$s=s_0+vt.$$

Thus, in uniform motion, the distance $s$ is a linear function of the time $t$, and the coefficient of $t$ is the speed: $v=(s-s_0)/t$.

## EXERCISES

**1.** If the constants $m$ and $b$ (§ 28) are given numerically, any number of points of the line can be located by arbitrarily assigning to the abscissa $x$ any series of values and computing from the function the corresponding values of the ordinates. This process is known as *plotting a line by points.* Two points are sufficient to determine the line.

Plot by points the following functions:

(*a*) $y = \frac{1}{2}x$, (*b*) $y = 2x - 5$, (*c*) $y = -3x + 5$,
(*d*) $y = -\frac{2}{3}x - 4$, (*e*) $y = x(x - 1)$, (*f*) $y = x^2$,
(*g*) $y = x^3$, (*h*) $y = 2^x$.

**2.** Draw the line represented by the equation $y = \frac{9}{5}x + 32$ of Example 2, § 29. What is its slope? What is the $y$-intercept? What is the meaning of each of these quantities if $y$ and $x$ represent the temperatures in Fahrenheit and in Centigrade measure, respectively?

**3.** Represent the equation $y = \frac{1}{12}x + 1$ of Example 3, § 29, by a figure. What is the meaning of the $y$-intercept?

**4.** Draw the line $s = s_0 + vt$ of Example 4, § 29, for the values $s_0 = 10$, $v = 3$. What is the meaning of $v$? Show that the speed $v$ may be thought of as the rate of increase of $s$ per second.

**5.** If, in the preceding exercise, $v$ be given a value greater than 3, how does the new line compare with the one just drawn?

**6.** If, in Ex. 4, $v$ is given the value 3, and $s_0$ several different values, show that the lines represented by the equation are parallel. Explain.

**7.** In experiments on the temperatures at various depths in a mine, the temperature (Centigrade) $T$ was found to be connected with the depth $d$ by the equation $T = 60 + .01\,d$, where $d$ is measured in feet. Draw a figure to represent this equation. Show that the rate of increase of the temperature was 1° per hundred feet.

**8.** In experiments on a pulley block, the pull $p$ (in lb.) required to lift a weight $w$ (in lb.) was found to be $p = .03\,w + 0.5$. Show that the rate of increase of $p$ is 3 lb. per hundredweight increase in $w$.

**9.** The velocity $v$ of a body falling from rest is proportional to the time: $v = gt$, where $g$ is a constant (about 32 in English units). If the body is *thrown* down with an initial velocity $v_0$, the velocity at any time $t$ is

$$v = v_0 + gt.$$

Draw a figure to represent this equation for $g = 32$, $v_0 = 10$. Show that $g$ is the *rate of increase of the velocity* (called the *acceleration*).

**30. General Linear Equation.** The equation

$$Ax + By + C = 0,$$

in which $A$, $B$, $C$ are any real numbers, is called the *general equation of the first degree* in $x$ and $y$. The coefficients $A$, $B$, $C$ are called the constants of the equation; $x$, $y$ are called the variables. It is assumed that $A$ and $B$ are not both zero. The terms $Ax$ and $By$ are of the first degree; the term $C$ is said to be of degree zero because it might be written in the form $Cx^0$; this term $C$ is also called the *constant term*.

*Every equation of the first degree,*

$$(5) \qquad \boldsymbol{Ax + By + C = 0},$$

*in which $A$ and $B$ are not both zero, represents a straight line; and conversely, every straight line can be represented by such an equation.* For this reason, every equation of the first degree is called a ***linear equation***.

The first part of this fundamental proposition follows from the fact that, when $B$ is not equal to zero, the equation can be reduced to the form $y = mx + b$ by dividing both sides by $B$; and we know that $y = mx + b$ represents a line (§ 25). When $B$ is equal to zero, the equation reduces to the form $x = a$, which also represents a line (§ 22).

The second part of the theorem follows from the fact that the equations which we have found in the preceding articles for any line are all particular cases of the equation

$$Ax + By + C = 0.$$

This equation still expresses the same relation between $x$ and $y$ when multiplied by any constant factor, not zero. Thus, any one of the constants $A$, $B$, $C$, if not zero, can be reduced to 1 by dividing both sides of the equation by this constant.

The equation is therefore said to contain only *two* (not three) *essential constants.*

**31. Conditions for Parallelism and for Perpendicularity.** It is easy to recognize whether two lines whose equations are $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ are parallel or perpendicular. The lines are parallel if they have the same slope, and they are perpendicular (§ 27) if the product of their slopes is equal to $-1$. The slopes of our lines are $-A/B$ and $-A'/B'$; hence these lines are *parallel* if $-A/B = -A'/B'$, *i.e.* if

$$A : B = A' : B';$$

and they are *perpendicular* if

$$\frac{A}{B} \cdot \frac{A'}{B'} = -1, \qquad \textit{i.e.} \text{ if} \qquad AA' + BB' = 0.$$

**32. Intercept Form.** If the constant term $C$ in a linear equation is zero, the equation represents a line through the origin. For, the coordinates $(0, 0)$ of the origin satisfy the equation

$$Ax + By = 0.$$

If the constant term $C$ is not equal to zero, the equation $Ax + By + C = 0$ can be divided by $C$; it then reduces to the form

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0.$$

If $A$ and $B$ are both different from zero, this can be written:

$$\frac{x}{-C/A} + \frac{y}{-C/B} = 1,$$

or putting $-C/A = a$, $-C/B = b$:

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{6}$$

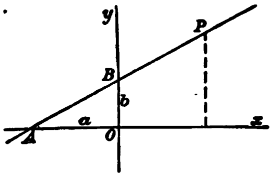

FIG. 25

The conditions $A \neq 0$, $B \neq 0$ mean evidently that the line is not parallel to either of the axes. Therefore the equation of any line, not passing through the origin, and not parallel to either axis, can be written in the form (6). With $y = 0$ this equation gives $x = a$; with $x = 0$ it gives $y = b$. Thus

$$a = -\frac{C}{A},\ b = -\frac{C}{B}$$

are the **intercepts** (Fig. 25) made by the line on the axes $Ox$, $Oy$, respectively (see § 25).

## EXERCISES

**1.** Write down the equations of the line whose intercepts on the axes $Ox$, $Oy$ are 5 and $-3$, respectively; the line whose intercepts are $-\frac{3}{2}$ and 7; the line whose intercepts are $-1$ and $-\frac{7}{2}$. Sketch each of the lines and reduce each of the equations to the form $Ax + By + C = 0$, so that $A$, $B$, $C$ are integers.

**2.** Find the intercepts of the lines: $3x - 2y = 1$, $x + 7y + 1 = 0$, $-3x + \frac{1}{2}y - 5 = 0$. Try to read off the values of the intercepts directly from these equations as they stand.

**3.** In Ex. 2, find the slopes of the lines.

**4.** Prove (6), § 32 by equality of areas, after clearing of fractions.

**5.** What is the equation of the axis $Oy$? of the axis $Ox$?

**6.** What is the value of $B$ such that the line represented by the equation $4x + By - 14 = 0$ passes through the point $(-5, 17)$.

**7.** What is the value of $A$ such that the line $Ax + 7y = 10$ has its $x$-intercept equal to $-8$?

**8.** Reduce each of the following equations to the intercept form (6), and draw the lines:

(a) $3x - 5y - 16 = 0$.
(b) $x + \frac{1}{2}y + 7 = 0$.
(c) $\dfrac{4x - 3y - 6}{x + y} = 2$.
(d) $5x = 3x + y - 10$.

**9.** Reduce the equations of Ex. 8 to the slope form (2), § 25.

**10.** Find the equation of the line of slope 6 passing through the point $(6, -5)$.

**11.** Show that the points $(-1, -7)$, $(\frac{1}{3}, -3)$, $(2, 2)$, $(-2, -10)$ lie on the same line.

**12.** Find the area of the triangle formed by the lines $x + y = 0$, $x - y = 0$, $x - a = 0$.

**13.** Show that the line $4(x - a) + 5(y - b) = 0$ is perpendicular to the line $5x - 4y - 10 = 0$ and passes through the point $(a, b)$.

**14.** A line has equal positive intercepts and passes through $(-5, 14)$. What is its equation? its slope?

**15.** If a line through the point $(6, 7)$ has the slope 4, what is its $y$-intercept? its $x$-intercept?

**16.** The Réaumur thermometer is graduated so that water freezes at 0° and boils at 80°. Draw the line that represents the reading $R$ of the Réaumur thermometer as a function of the corresponding reading $C$ of the Centigrade thermometer.

**17.** Express the value of a note of $1000 at the end of the first year as a function of the rate of interest. At 6 % simple interest its value is what function of the time in years?

**33. Line through One Point.** To find the line of given slope $m_1$ through a given point $P_1(x_1, y_1)$, observe that the equation must be of the form (2), viz.

$$y = m_1x + b,$$

since this line has the slope $m_1$. If this line is to pass through the given point, the coordinates $x_1$, $y_1$ must satisfy this equation, *i.e.* we must have

$$y_1 = m_1x_1 + b.$$

This equation determines $b$, and the value of $b$ so found might be substituted in the preceding equation. But we can eliminate $b$ more readily between the two equations by subtracting the latter from the former. This gives

$$y - y_1 = m_1(x - x_1)$$

as *the equation of the line of slope $m_1$ through $P_1(x_1, y_1)$*.

The problem of finding *a line through a given point parallel, or perpendicular, to a given line* is merely a particular case of the problem just solved, since the slope of the required line can be found from the equation of the given line (§ 27). If the slope of the given line is $m_1 = \tan \alpha_1$, the slope of any parallel line is also $m_1$, and the slope of any line perpendicular to it is

$$m_2 = \tan(\alpha_1 + \tfrac{1}{2}\pi) = -\cot \alpha_1 = -\frac{1}{m_1}.$$

**34. Line through Two Points.** To find the line through two given points, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, observe (Fig. 26) that the slope of the required line is evidently

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

if, as in § 9, we denote by $\Delta x$, $\Delta y$ the projections of $P_1P_2$ on $Ox$, $Oy$;

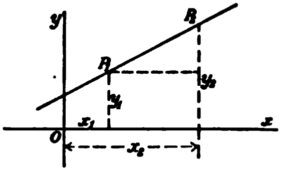

FIG. 26

and as the line is to pass through $(x_1, y_1)$, we find its equation by § 33 as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

or

$$y - y_1 = \frac{\Delta y}{\Delta x}(x - x_1).$$

The equation of the line through two given points $(x_1, y_1)$, $(x_2, y_2)$ can also be written in the *determinant form*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

which (§ 14) means that the point $(x, y)$ is such as to form with the given points a triangle of zero area. By expanding the determinant it can be shown that this equation agrees with the preceding equation.

### EXERCISES

**1.** Find the equation of the line through the point $(-7, 2)$ parallel to the line $y = 3x$.

**2.** Show that the points $(4, -3)$, $(-5, 2)$, $(5, 20)$ are the vertices of a right triangle.

**3.** Find the equation of the line through the point $(-6, -3)$ which makes an angle of 30° with the axis $Ox$; 30° with the axis $Oy$.

**4.** Does the line of slope $\frac{7}{9}$ through the point $(4, 3)$ pass through the point $(-5, -4)$?

**5.** Find the equation of the line through the point $(-2, 1)$ parallel to the line through the points $(4, 2)$ and $(-3, -2)$.

**6.** Find the equations of the lines through the origin which trisect that portion of the line $5x - 6y = 60$ which lies in the fourth quadrant.

**7.** What are the intercepts of the line through the points $(2, -3)$, $(-5, 4)$?

**8.** Show that the equation of the line through the point $(a, b)$ perpendicular to the line $Ax + By + C = 0$ is $(x - a)/A = (y - b)/B$.

**9.** Find the equations of the diagonals of the rectangle formed by the lines $x + a = 0$, $x - b = 0$, $y + c = 0$, $y - d = 0$.

**10.** Find the equation of the perpendicular bisector of the line joining the points $(4, -5)$ and $(-3, 2)$. Show that any point on it is equally distant from each of the two given points.

**11.** Find the equation of the line perpendicular to the line $4x-3y+6=0$ that passes through the midpoint of $(-4, 7)$ and $(2, 2)$.

**12.** What are the coordinates of a point equidistant from the points $(2, -3)$ and $(-5, 0)$ and such that the line joining the point to the origin has a slope 1?

**13.** In an experiment with a pulley-block it is assumed that the relation between the load $l$ and the pull $p$ required to lift it is linear. Find the relation if $p = 8$ when $l = 100$, and $p = 12$ when $l = 200$.

**14.** In an experiment in stretching a brass wire it is assumed that the elongation $E$ is connected with the tension $t$ by means of a linear relation. Find this relation if $t = 18$ lb. when $E = .1$ in., and $t = 58$ lb. when $E = .3$ in.

# CHAPTER III

## RELATIONS BETWEEN TWO OR MORE LINES

**35. Intersection of Two Lines.** *The point of intersection of any two lines is found by solving the equations of the lines as simultaneous equations.* For, the coordinates of the point of intersection must satisfy each of the two equations, since this point lies on each of the two lines; and it is the only point having this property. Thus, by solving the equations

$$4x - 3y + 3 = 0,$$
$$3x + 5y - 34 = 0,$$

we find $x = 3$, $y = 5$; hence $(3, 5)$ is the point of intersection of the two lines represented by these equations.

**36. Particular Cases.** The equations of any two lines being given, say

$$(1) \qquad \begin{aligned} a_1x + b_1y &= k_1, \\ a_2x + b_2y &= k_2, \end{aligned}$$

we find by the usual method, that is, first multiplying by $b_2$, $b_1$ and subtracting, then multiplying by $a_2$, $a_1$ and subtracting:

$$(2) \qquad \begin{aligned} (a_1b_2 - a_2b_1)x &= k_1b_2 - k_2b_1, \\ (a_1b_2 - a_2b_1)y &= a_1k_2 - a_2k_1. \end{aligned}$$

The expression $a_1b_2 - a_2b_1$ is called the *determinant of the equations.* Two cases must be distinguished according as this determinant is $\neq 0$ or $= 0$.

(*a*) If $a_1b_2 - a_2b_1 \neq 0$, which means by § 31 that the lines are not parallel, we can divide the equations (2) by this determinant and thus find $x$ and $y$. If, in particular, $k_1$ and $k_2$ are both zero, that is, if the equations (1) are *homogeneous* and hence represent two lines through the origin, we find from (2) $x = 0$ and $y = 0$, as was to be expected.

(*b*) If $a_1b_2 - a_2b_1 = 0$, that is, if the lines (1) are parallel, we cannot in (2) divide by $a_1b_2 - a_2b_1$; the equations (2) then become

$$0 \cdot x = k_1b_2 - k_2b_1,$$
$$0 \cdot y = a_1k_2 - a_2k_1,$$

and cannot be satisfied by any values of $x$ and $y$ unless the right-hand members are both zero. In the latter case we have

$$\frac{k_1}{k_2} = \frac{a_1}{a_2} = \frac{b_1}{b_2};$$

that is, the second equation is merely a multiple of the first. In this case the two equations (1) represent the same line and have therefore all points in common.

## EXERCISES

**1.** Find the coordinates of the points of intersection of the following lines; and check by a sketch:

(*a*) $\begin{cases} 5x - 7y + 11 = 0, \\ 3x + 2y - 12 = 0. \end{cases}$ (*b*) $\begin{cases} 3x + 2y = 0, \\ 6x - 4y + 4 = 0. \end{cases}$ (*c*) $\begin{cases} 2.4x + 3.1y = 4.5, \\ .8x + 2y = 6.2. \end{cases}$

**2.** Do the following pairs of lines intersect, or are they parallel or coincident?

(*a*) $\begin{cases} 3x - 6y - 8 = 0, \\ x - 2y + 1 = 0. \end{cases}$ (*b*) $\begin{cases} 2x - 6y - 4 = 0, \\ x - 3y - 2 = 0. \end{cases}$ (*c*) $\begin{cases} x + \frac{1}{2}y = 0, \\ 2x + 3y = 0. \end{cases}$

**3.** Show that the condition that the three lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, $A''x + B''y + C'' = 0$ meet at a point is

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

**4.** Show that the straight lines $3x + y - 1 = 0$, $x - 3y + 13 = 0$, $2x - y + 6 = 0$ have a common point.

5. Show that the lines joining the midpoints of the sides of any triangle divide the triangle into four equal triangles.

6. Show that the altitudes of any triangle meet in a point.

7. Show that the medians of any triangle meet in a point.

8. Show that the line through the origin perpendicular to the line through the points $(a, 0)$ and $(0, b)$ meets the lines through the points $(a, 0)$, $(-b, b)$ and $(0, b)$, $(a, -a)$ in a common point.

**37. Angle between Two Lines.** We shall understand by the angle $(l, l') = \theta$ between two lines $l$ and $l'$ the least angle through which $l$ must be turned counterclockwise about the point of intersection to come to coincidence with $l'$. This angle $\theta$ is equal to the difference of the slope angles $\alpha$, $\alpha'$ (Fig. 27) of the two lines. Thus, if $\alpha' > \alpha$, we have $\theta = \alpha' - \alpha$, since $\alpha'$ is the exterior angle of a triangle, two of whose interior angles are $\alpha$ and $\theta$. It follows that

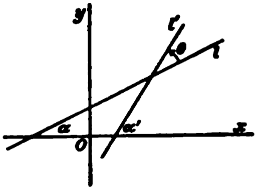

FIG. 27

$$\tan\theta = \tan(\alpha' - \alpha) = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha\tan\alpha'}. \tag{3}$$

If the equations of $l$ and $l'$ are

$$y = mx + b,\quad y = m'x + b',$$

respectively, we have $\tan\alpha = m$, $\tan\alpha' = m'$; hence

$$\tan\theta = \frac{m' - m}{1 + mm'}. \tag{4}$$

If the equations of $l$ and $l'$ are

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

respectively, we have $\tan\alpha = -A/B$, $\tan\alpha' = -A'/B'$; hence

$$\tan\theta = \frac{AB' - A'B}{AA' + BB'}. \tag{5}$$

**38.** It follows, in particular, that the two lines $l$ and $l'$, § 37, are ***parallel*** if and only if

$$m' = m, \qquad \text{or } AB' - A'B = 0;$$

and they are ***perpendicular*** to each other if and only if

$$m' = -\frac{1}{m}, \text{ or } AA' + BB' = 0.$$

(Compare §§ 27, 31.) Hence, to write down the equation of a line *parallel* to a given line, replace the constant term by an arbitrary constant; to write down the equation of a line *perpendicular* to a given line, interchange the coefficients of $x$ and $y$, changing the sign of one of them, and replace the constant term by an arbitrary constant.

### EXERCISES

**1.** Determine whether the following pairs of lines are parallel or perpendicular: $3x + 2y - 6 = 0$, $2x - 3y + 4 = 0$; $5x + 3y - 6 = 0$, $10x + 6y + 2 = 0$; $2x + 5y - 14 = 0$, $8x - 3y + 6 = 0$.

**2.** Find the point of intersection of the line $5x + 8y + 17 = 0$ with its perpendicular through the origin.

**3.** Find the point of intersection of the lines through the points $(6, -2)$ and $(0, 2)$, and $(4, 5)$ and $(-1, -4)$.

**4.** Find the perpendicular bisector of the line-segment joining the point $(3, 4)$ to the point of intersection of the lines $2x - y + 1 = 0$ and $3x + y - 16 = 0$.

**5.** Find the lines through the point of intersection of the lines $5x - y = 0$, $x + 7y - 9 = 0$ and perpendicular to them.

**6.** Find the area of the triangle formed by the lines $3x + 4y = 8$, $6x - 5y = 30$, and $x = 0$.

**7.** Find the area of the triangle formed by the lines $x + y - 1 = 0$, $2x + y + 5 = 0$, and $x - 2y - 10 = 0$.

**8.** Find the point of intersection of the lines

$$(a)\quad \frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1.$$

$$(b)\quad \frac{x}{a} + \frac{y}{b} = 1, \quad y = mx + b.$$

**9.** Find the area of the triangle formed by the lines $y = m_1x + b_1$, $y = m_2x + b_2$ and the axis $Ox$.

**10.** The vertices of a triangle are $(5, -4)$, $(-3, 2)$, $(7, 6)$. Find the equations of the medians and their point of intersection.

**11.** Find the angle between the lines $4x-3y-6=0$ and $x-7y+6=0$.

**12.** Find the tangent of the angle between the lines (*a*) $4x-3y+6=0$ and $9x+2y-8=0$; (*b*) $3x+6y-11=0$ and $x+2y-3=0$.

**13.** Find the two lines through the point (6, 10) inclined at 45° to the line $3x-2y-12=0$.

**14.** Find the lines through the point $(-3, 7)$ such that the tangent of the angle between each of these lines and the line $6x-2y+11=0$ is $\frac{1}{3}$.

**15.** Show that the angle between the lines $Ax+By+C=0$ and $(A+B)x-(A-B)y+D=0$ is 45°.

**16.** Find the lines which make an angle of 45° with the line $4x-7y+6=0$ and bisect the portion of it intercepted by the axes.

**17.** The hypotenuse of an isosceles right-angled triangle lies on the line $3x-6y-17=0$. The origin is one vertex; what are the others?

**39. Polar Equation of Line.** The position of a line in the plane is fully determined by the length $p = ON$ (Fig. 28) of the perpendicular let fall from the origin on the line and the angle $\beta = xON$ made by this perpendicular with the axis $Ox$.

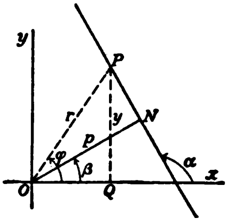

FIG. 28

Then $p$ and $\beta$ are evidently the *polar* coordinates of the point $N$ (§ 16). Let $P$ be any point of the line and $OP = r$, $xOP = \phi$ its polar coordinates. As the projection of $OP$ on the perpendicular $ON$ is equal to $ON$, and the angle $NOP = \phi - \beta$, we have

$$(6) \qquad r\cos(\phi - \beta) = p.$$

This is the *equation of the line NP in polar coordinates.*

**40. Normal Form.** The last equation can be transformed to Cartesian coordinates by expanding the cosine:

$$r \cos \phi \cos \beta + r \sin \phi \sin \beta = p$$

and observing (§ 17) that $r \cos \phi = x$, $r \sin \phi = y$; the equation then becomes

$$x \cos \beta + y \sin \beta = p. \tag{7}$$

This equation, which is called the ***normal form*** of the equation of the line, can be read off directly from the figure; it means that the sum of the projections of $x$ and $y$ on the perpendicular to the line is equal to the projection of $r$ (§ 20).

Observe that in the normal form (7) the number $p$ is always positive, being the distance of the line from the origin, or the radius vector of the point $N$. Hence $x \cos \beta + y \sin \beta$ is always positive; this also appears by considering that $x \cos \beta + y \sin \beta$ is the projection of the radius vector $OP$ on $ON$, and that this radius vector makes with $ON$ an angle that cannot be greater than a right angle.

The angle $\beta = xON$ is, as a polar angle (§ 16), always understood to be the angle through which the axis $Ox$ must be turned counterclockwise about the origin to make it coincide with $ON$; it can therefore have any value from 0 to $2\pi$. By drawing the parallel to the line $NP$ through the origin it is readily seen that, if $\alpha$ is the slope angle of the line $NP$, we have

$$\beta = \alpha + \tfrac{1}{2}\pi \quad \text{or} \quad \beta = \alpha + \tfrac{3}{2}\pi$$

according as the line lies on one side of the origin or the other, angles differing by $2\pi$ being regarded as equivalent. Thus, in Fig. 28, $\alpha = 120°$, $\beta = \alpha + \frac{3}{2}\pi = 120° + 270° = 390°$, which is equivalent to $30°$. For a parallel on the opposite side of the origin we should have $\beta = \alpha + \frac{1}{2}\pi = 120° + 90° = 210°$.

**41. Reduction to Normal Form.** The equation

$$Ax+By+C=0$$

is in general not of the form (7), since in the latter equation the coefficients of $x$ and $y$, being the cosine and sine of an angle, have the property that the sum of their squares is equal to 1, while in the former equation the sum of the squares of $A$ and $B$ is in general not equal to 1. But the general equation

$$Ax+By+C=0$$

can be reduced to the normal form (7) by multiplying it by a factor $k$ properly chosen; we know (§ 30) that the equation

$$kAx+kBy+kC=0$$

represents the same line as does the equation $Ax+By+C=0$. Now if we select $k$ so that

$$kA=\cos\beta,\quad kB=\sin\beta,\quad kC=-p,$$

the equation $Ax+By+C=0$ reduces to the normal form $x\cos\beta+y\sin\beta-p=0$. The first two conditions give

$$k^2A^2+k^2B^2=\cos^2\beta+\sin^2\beta=1,$$

whence $$k=\pm\frac{1}{\sqrt{A^2+B^2}}.$$

Since the right-hand member $p$ in the normal form (7) is positive, the sign of the square root must be selected so that $kC$ becomes negative. We have therefore the rule:

*To reduce the general equation $Ax+By+C=0$ to the normal form*

$$x\cos\beta+y\sin\beta-p=0,$$

*divide by $-\sqrt{A^2+B^2}$ when $C$ is positive and by $+\sqrt{A^2+B^2}$ when $C$ is negative.*

Then the coefficients of $x$ and $y$ will be $\cos\beta$, $\sin\beta$, respectively, and the constant term will be the distance $p$ of the line from the origin.

Thus, to reduce $3x+2y+5=0$ to the normal form, divide by $-\sqrt{3^2+2^2}=-\sqrt{13}$; this gives

$$\cos\beta=-\frac{3}{\sqrt{13}},\ \sin\beta=-\frac{2}{\sqrt{13}},\ -p=-\frac{5}{\sqrt{13}};$$

*i.e.* the normal form is

$$-\frac{3}{\sqrt{13}}x-\frac{2}{\sqrt{13}}y=\frac{5}{\sqrt{13}}.$$

The perpendicular to the line from the origin has the length $5/\sqrt{13}$; and as both $\cos\beta$ and $\sin\beta$ are negative, this perpendicular lies in the third quadrant. Draw the line.

Reduce the equation $3x+2y-5=0$ to the normal form.

**42. Distance of a Point from a Line.** If, in Fig. 28, we take instead of a point $P$ *on* the line any point $P_1$ $(x_1, y_1)$ *not* on the line (Fig. 29), the expression $x_1\cos\beta+y_1\sin\beta$ is still the projection on $ON$ (produced if necessary) of the radius vector $OP_1$. But this projection $OS$ differs from the normal $ON=p$ to the line. The figure shows that the difference

$$x_1\cos\beta+y_1\sin\beta-p=OS-ON=NS$$

is equal to the distance $N_1P_1$ of the point $P_1$ *from* the line.

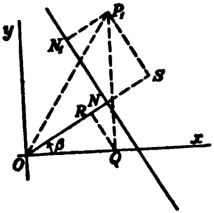

FIG. 29

Thus, to find the distance of any point $P_1$ $(x_1, y_1)$ from a line whose equation is given in the normal form

$$x\cos\beta+y\sin\beta-p=0,$$

it suffices to substitute in the left-hand member of this equation for $x$, $y$ the coordinates $x_1$, $y_1$ of the point $P_1$. The expression

$$x_1\cos\beta+y_1\sin\beta-p$$

then represents *the distance of* $P_1$ *from the line.*

If this expression is negative, the point $P_1$ lies on the same side of the line as does the origin; if it is positive, the point

$P_1$ lies on the opposite side of the line. Any line thus divides the plane into two regions which we may call the positive and negative regions; that in which the origin lies is the negative region.

To find the distance of a point $P_1$ $(x_1, y_1)$ from a line given in the general form

$$Ax + By + C = 0,$$

we have only to reduce the equation to the normal form (§ 41) and then apply the rule given above. Thus the distance is

$$\frac{Ax_1 + By_1 + C}{-\sqrt{A^2 + B^2}} \quad \text{or} \quad \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}},$$

according as $C$ is positive or negative.

**43. Bisector of an Angle.** To find the ***bisectors*** of the angles between two lines given in the normal form

$$x \cos \beta + y \sin \beta - p = 0,$$
$$x \cos \beta' + y \sin \beta' - p' = 0,$$

observe that for any point on either bisector its distances from the two lines must be equal in absolute value. Hence the equations of the bisectors are

$$x \cos \beta + y \sin \beta - p = \pm (x \cos \beta' + y \sin \beta' - p').$$

To distinguish the two bisectors, observe that for the bisector of that pair of vertical angles which contains the origin (Fig. 30) the perpendicular distances are, in one angle both positive, in the other both negative; hence the plus sign gives this bisector.

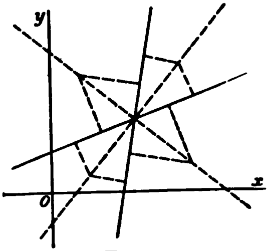

FIG. 30

If the equations of the lines are given in the general form

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

first reduce the equations to the normal form, and then apply the previous rule.

## EXERCISES

**1.** Draw the lines represented by the following equations:

(*a*) $r\cos(\phi - \frac{1}{6}\pi) = 6.$ (*e*) $r\cos(\phi + \frac{2}{3}\pi) = 3.$

(*b*) $r\cos(\phi - \pi) = 4.$ (*f*) $r\sin(\phi - \frac{1}{6}\pi) = 8.$

(*c*) $r\cos\phi = 10.$ (*g*) $r\sin(\phi + \frac{3}{4}\pi) = 7.$

(*d*) $r\sin\phi = 5.$ (*h*) $r\cos(\phi - \frac{3}{4}\pi) = 0.$

**2.** In polar coordinates, find the equations of the lines: (*a*) parallel to and at the distance 5 from the polar axis (above and below); (*b*) perpendicular to the polar axis and at the distance 4 from the pole (to the right and left); (*c*) inclined at an angle of $\frac{1}{3}\pi$ to the polar axis and at the distance 12 from the pole.

**3.** Express in polar coordinates the sides of the rectangle $OABC$ if $OA = 6$ and $AB = 9$, $OA$ being taken as polar axis.

**4.** What lines are represented by (7) when $p$ is constant, while $\beta$ varies from zero to $2\pi$? What lines when $p$ varies while $\beta$ remains constant?

**5.** The perpendicular from the origin to a line is 5 units in length and makes an angle $\tan^{-1}\frac{5}{12}$ with the axis $Ox$. Find the equation of the line.

**6.** Reduce the equations of Ex. 8, p. 33, to the normal form (7).

**7.** Find the equations of the lines whose slope angle is 150° and which are at the distance 4 from the origin.

**8.** What is the equation of the line through the point $(-3, 5)$ whose perpendicular from the origin makes an angle of 120° with the axis $Ox$?

**9.** For the line $7x - 24y - 20 = 0$ find the intercepts, slope, length of perpendicular from the origin and the sine and cosine of the angle which this perpendicular makes with the axis $Ox$.

**10.** Find by means of $\sin\beta$ and $\cos\beta$ the quadrants crossed by the line $4x - 5y = 8$.

**11.** Put the following equations in the form (7) and thus find $p$, $\sin\beta$, $\cos\beta$:

(*a*) $y = mx + b.$ (*b*) $\frac{x}{a} + \frac{y}{b} = 1.$ (*c*) $3x = 4y.$

**12.** Is the point $(3, -4)$ on the positive or negative side of the line through the points $(-5, 2)$ and $(4, 7)$?

**13.** Is the point $(-1, -\frac{4}{3})$ on the positive or negative side of the line $4x - 9y - 8 = 0$?

**14.** Find by means of an altitude and a side the area of the triangle formed by the lines $3x + 2y + 10 = 0$, $4x - 3y + 16 = 0$, $2x + y - 4 = 0$. Check the result with another altitude and side.

**15.** Find the distance between the parallel lines (*a*) $3x - 5y - 4 = 0$ and $6x - 10y + 7 = 0$; (*b*) $5x + 7y + 9 = 0$ and $15x + 21y - 3 = 0$.

**16.** What is the length of the perpendicular from the origin to the line through the point $(-5, -4)$ whose slope angle is 60°?

**17.** What are the equations of the lines whose distances from the origin are 6 units each and whose slopes are $\frac{5}{3}$?

**18.** Find the points on the axis $Ox$ whose perpendicular distances from the line $24x - 7y - 16 = 0$ are $\pm 5$.

**19.** Find the point equidistant from the points $(4, -3)$ and $(-2, 1)$, and at the distance 4 from the line $3x - 4y - 5 = 0$.

**20.** Find the line parallel to $12x - 5y - 6 = 0$ and at the same distance from the origin; farther from the origin by a distance 3.

**21.** Find the two lines through the point $(1, \frac{20}{3})$ such that the perpendiculars let fall from the point $(6, 5)$ are of length 5.

**22.** Find the line perpendicular to $4x - 7y - 10 = 0$ which crosses the axis $Ox$ at a distance 6 from the point $(-2, 0)$.

**23.** Find the bisectors of the angles between the lines: (*a*) $x - y - 4 = 0$ and $3x + 3y + 7 = 0$; (*b*) $5x - 12y - 16 = 0$ and $24x + 7y + 60 = 0$.

**24.** Find the bisectors of the angles of the triangle formed by the lines $5x + 12y + 20 = 0$, $4x - 3y - 6 = 0$, $3x - 4y + 5 = 0$ and the center of the circle inscribed in the triangle.

**25.** Find the bisector of that angle between the lines $3x - \sqrt{3}y + 10 = 0$, $\sqrt{2}x + y - 6 = 0$ in which the origin lies.

**26.** If two lines are given in the normal form, what is represented by their sum and what by their difference?

**27.** Show that the angle between the lines $x + y = 0$ and $x - y = 0$ is 90° whether the axes are rectangular or oblique.

**44. Pencils of Lines.** All lines through one and the same point are said to form a ***pencil;*** the point is called the ***center*** of the pencil. If

$$(8)\qquad \begin{cases} Ax + By + C = 0, \\ A'x + B'y + C' = 0 \end{cases}$$

are any two different lines of a pencil, the equation

$$(9)\qquad Ax + By + C + k(A'x + B'y + C') = 0,$$

where $k$ is any constant, represents a line of the pencil. For, the equation (9) is of the first degree in $x$ and $y$, and the coefficients of $x$ and $y$ cannot both be zero, since this would mean that the lines (8) are parallel. Moreover, the line (9) passes through the center of the pencil (8) because the coordinates of the point that satisfies each of the equations (8) also satisfy the equation (9).

All lines parallel to the same direction are said to form a *pencil of parallels.* It is readily seen that if the lines (8) are parallel, the equation (9) represents a line parallel to them.

### EXERCISES

1. Find the line: (*a*) through the point of intersection of the lines $4x - 7y + 5 = 0$, $6x + 11y - 7 = 0$ and the origin; (*b*) through the point of intersection of the lines $4x - 2y - 3 = 0$, $x + y - 5 = 0$ and the point $(-2, 3)$; (*c*) through the point of intersection of the lines $4x - 5y + 6 = 0$, $y - x - 3 = 0$, of slope 3; (*d*) through the intersection of $5x - 6y + 10 = 0$, $2x + 3y - 12 = 0$, perpendicular to $4y + x = 0$.

2. Find the line of the pencil $x - 5 = 0$, $y + 2 = 0$ that is inclined to the axis $Ox$ at 30°.

3. Determine the constant $b$ of the line $y = 3x + b$ so that this line shall belong to the pencil $3x - 4y + 6 = 0$, $x = 5$.

4. Find the line joining the centers of the pencils $x - 3y = 12$, $5x - 2y = 1$ and $x + y = 6$, $4x - 5y = 3$.

5. Find the line of the pencil $4x - 5y - 12 = 0$, $3x + 2y - 16 = 0$ that makes equal intercepts on the axes.

**45. Non-linear Equations representing Lines.** When two lines are given, say

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

then the equation

$$(Ax + By + C)(A'x + B'y + C') = 0,$$

obtained by multiplying the left-hand members (the right-hand members being reduced to zero) is satisfied by all the points of the first given line as well as all the points of the second given line, and by no other points.

The product equation which is of the second degree is therefore said to represent the two given lines. Similarly, by equating to zero the product of the left-hand members of the equations of three or more straight lines (whose right-hand members are zero) we find a single equation representing all these lines. An equation of the $n$th degree *may* therefore represent $n$ straight lines, viz. when its left-hand member (the right-hand member being zero) can be resolved into $n$ linear factors, with real coefficients.

### EXERCISES

**1.** Find the common equation of the two axes of coordinates.

**2.** Show that $n$ lines through the origin are represented by a *homogeneous* equation (*i.e.* one in which all terms are of the same degree in $x$ and $y$) of the $n$th degree.

**3.** Draw the lines represented by the following equations:

(*a*) $(x - a)(y - b) = 0.$

(*b*) $3x^2 - xy - 4y^2 = 0.$

(*c*) $x^2 - 9y^2 = 0.$

(*d*) $ax^2 + by^2 = 0.$

(*e*) $x^2 - x - 12 = 0.$

(*f*) $xy - ax = 0.$

(*g*) $y^3 - 5y^2 + 6y = 0.$

(*h*) $x^3y - xy = 0.$

(*i*) $y^3 - 6xy^2 + 11x^2y - 6x^3 = 0.$

**4.** What relation must hold between $a$, $h$, $b$, if the lines represented by $ax^2 + 2hxy + by^2 = 0$ are to be real and distinct, coincident, imaginary?

## MISCELLANEOUS EXERCISES

**1.** Find the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$. What is the condition for these lines to be perpendicular? coincident?

**2.** Reduce the general equation $Ax + By + C = 0$ to the normal form $x \cos \beta + y \sin \beta = p$ by considering that, if both equations represent the same line, the intercepts must be the same.

**3.** Find the line through $(x_1, y_1)$ making equal intercepts on the axes.

**4.** Find the area of the triangle formed by the lines $y = m_1x + b_1$, $y = m_2x + b_2$, $y = b$.

**5.** What does the equation $\phi =$ const. represent in polar coordinates?

**6.** Find the polar equation of the line through $(6, \pi)$ and $(4, \frac{1}{2}\pi)$.

**7.** Derive the determinant expression for the area of a triangle (§ 14) by multiplying one side by half the altitude.

**8.** The weights $w$, $W$ being suspended at distances $d$, $D$, respectively, from the fulcrum of a lever, we have by the *law of the lever* $WD = wd$. If the weights are shifted along the lever, then to every value of $d$ corresponds a definite value of $D$; *i.e.* $D$ is a function of $d$. Represent this function graphically; interpret the part of the line in the third quadrant.

**9.** A train, after leaving the station $A$, attains in the first 6 minutes, $1\frac{1}{2}$ miles from $A$, the speed of 30 miles per hour with which it goes on. How far from $A$ will it be 50 minutes after starting? (Compare Example 4, § 29.) Illustrate graphically, taking $s$ in miles, $t$ in minutes.

**10.** A train leaves Detroit at 8 hr. 25 m. A.M. and reaches Chicago at 4 hr. 5 m. P.M.; another train leaves Chicago at 10 hr. 30 m. A.M. and arrives in Detroit at 5 hr. 30 m. P.M. The distance is 284 miles. Regarding the motion as uniform and neglecting the stops, find graphically and analytically where and when the trains meet. If the scale of distances (in miles) be taken 1/20 of the scale of times (in hours), how can the velocities be found from the slopes?

**11.** A stone is dropped from a balloon ascending vertically at the rate of 24 ft./sec.; express the velocity as a function of the time (Example 5, § 29). What is the velocity after 4 sec.?

**12.** How long will a ball rise if thrown vertically upward with an initial velocity of 100 ft./sec.?

# CHAPTER IV

## THE CIRCLE

**46. Circles.** A circle, in a given plane, is defined as *the locus of all those points of the plane which are at the same distance from a fixed point.*

Let $C\ (h, k)$ be the center, $r$ the radius (Fig. 31); the necessary and sufficient condition that any point $P\ (x, y)$ is at the distance $r$ from $C\ (h, k)$ is that

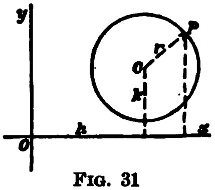

FIG. 31

$$(1)\qquad (x-h)^2+(y-k)^2=r^2.$$

This equation, which is satisfied by the coordinates $x$, $y$ of every point on the circle, and by the coordinates of no other point, is called *the equation of the circle of center $C\ (h, k)$ and radius $r$.*

If the center of the circle is at the origin $O\ (0, 0)$, the equation of the circle is evidently

$$(2)\qquad x^2+y^2=r^2.$$

### EXERCISES

Write down the equations of the following circles:

(*a*) center (3, 2), radius 7;
(*b*) center at origin, radius 3;
(*c*) center at $(-a, 0)$, radius $a$;
(*d*) circle of any radius touching the axis $Ox$ at the origin;
(*e*) circle of any radius touching the axis $Oy$ at the origin.

Illustrate each case by a sketch.

**47. Equation of Second Degree.** Expanding the equation (1) of § 46, we obtain the equation of the circle in the new form

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is an *equation of the second degree* in $x$ and $y$. But it is of a particular form. ***The general equation of the second degree*** in $x$ and $y$ is of the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0; \tag{3}$$

*i.e.* it contains a constant term, $C$; two terms of the first degree, one in $x$ and one in $y$; and three terms of the second degree, one in $x^2$, one in $xy$, and one in $y^2$.

If in this general equation we have

$$H = 0, \ B = A \neq 0,$$

it reduces, upon division by $A$, to the form

$$x^2 + y^2 + \frac{2G}{A}x + \frac{2F}{A}y + \frac{C}{A} = 0,$$

which agrees with the form (1) of the equation of a circle, except for the notation for the coefficients.

We can therefore say that *any equation of the second degree which contains no xy-term and in which the coefficients of $x^2$ and $y^2$ are equal, may represent a circle.*

**48. Determination of Center and Radius.** To draw the circle represented by the general equation

$$\boldsymbol{Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,} \tag{4}$$

where $A$, $G$, $F$, $C$ are any real numbers while $A \neq 0$, we first divide by $A$ and *complete the squares in $x$ and $y$; i.e.* we first write the equation in the form

$$\left(x + \frac{G}{A}\right)^2 + \left(y + \frac{F}{A}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}.$$

The left-hand member represents the square of the distance of the point $(x, y)$ from the point $(-G/A, -F/A)$; the right-hand member is constant. The given equation therefore represents the circle whose center has the coordinates

$$h = -\frac{G}{A},\ k = -\frac{F}{A},$$

and whose radius is

$$r = \sqrt{\frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}} = \frac{1}{A}\sqrt{G^2 + F^2 - AC}.$$

This radius is, however, imaginary if $G^2 + F^2 < AC$; in this case the equation is not satisfied by any points with real coordinates.

If $G^2 + F^2 = AC$, the radius is zero, and the equation is satisfied only by the coordinates of the point $(-G/A, \ -F/A)$.

If $G^2 + F^2 > AC$, the radius is real, and the equation represents a real circle.

Thus, *the general equation of the second degree* (3), § 47, *represents a circle if, and only if,*

$$A = B \neq 0,\ H = 0,\ G^2 + F^2 > AC.$$

**49. Circle determined by Three Conditions.** The equation (1) of the circle contains three constants $h, k, r$. The general equation (4) contains four constants of which, however, only three are essential since we can always divide through by one of these constants. Thus, dividing by $A$ and putting $2\,G/A = a$, $2\,F/A = b$, $C/A = c$, the general equation (4) assumes the form

$$x^2 + y^2 + ax + by + c = 0, \tag{5}$$

with the three constants $a, b, c$.

The existence of three constants in the equation corresponds to the possibility of determining a circle geometrically, in a variety of ways, by three conditions. It should be remembered in this connection that the equation of a straight line contains two essential constants, the line being determined by two geometrical conditions (§ 30).

## EXERCISES

**1.** Draw the circles represented by the following equations:

(*a*) $2x^2 + 2y^2 - 8x + 5y + 1 = 0.$ (*b*) $3x^2 + 3y^2 + 17x - 15y - 6 = 0.$
(*c*) $4x^2 + 4y^2 - 6x - 10y + 4 = 0.$ (*d*) $x^2 + y^2 + x - 4y = 0.$
(*e*) $2x^2 + 2y^2 - 7x = 0.$ (*f*) $x^2 + y^2 - 3x - 6 = 0.$

**2.** What is the equation of the circle of center $(h, k)$ that touches the axis $Ox$? that touches the axis $Oy$? that passes through the origin?

**3.** What is the equation of any circle whose center lies on the axis $Ox$? on the axis $Oy$? on the line $y = x$? on the line $y = 2x$? on the line $y = mx$?

**4.** Find the equation of the circle whose center is at the point $(-4, 6)$ and which passes through the point $(2, 0)$.

**5.** Find the circle that has the points $(4, -3)$ and $(-2, -1)$ as ends of a diameter.

**6.** A swing moving in the vertical plane of the observer is 48 ft. away and is suspended from a pole 27 ft. high. If the seat when at rest is 2 ft. above the ground, what is the equation of the path (for the observer as origin)? What is the distance of the seat from the observer when the rope is inclined at 45° to the vertical?

**7.** Find the locus of a point whose distance from the point $(a, b)$ is $\kappa$ times its distance from the origin.

Let $P(x, y)$ be *any* point of the locus; then the condition is

$$\sqrt{(x-a)^2 + (y-b)^2} = \kappa\sqrt{x^2 + y^2};$$

upon squaring and rearranging this becomes:

$$(1-\kappa^2)x^2 + (1-\kappa^2)y^2 - 2ax - 2by + a^2 + b^2 = 0.$$

Hence for any value of $\kappa$ except $\kappa = 1$, the locus is a circle whose center is $a/(1-\kappa^2)$, $b/(1-\kappa^2)$ and whose radius is $\kappa\sqrt{a^2+b^2}/(1-\kappa^2)$. What is the locus when $\kappa = 1$?

**8.** Find the locus of a point twice as far from the origin as from the point $(6, -3)$. Sketch.

**9.** What is the locus of a point whose distances from two points $P_1$, $P_2$ are in the constant ratio $\kappa$?

**10.** Determine the locus of the points which are $\kappa$ times as far from the point $(-2, 0)$ as from the point $(2, 0)$. Assign to $\kappa$ the values $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$, $\frac{1}{5}\sqrt{5}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{2}\sqrt{2}$ and illustrate with sketches drawn with respect to the same axes.

**11.** Determine the locus of a point whose distance from the line $3x - 4y + 1 = 0$ is equal to the square of its distance from the origin. Illustrate with a sketch.

**12.** Determine the locus of a point if the square of its distance from the line $x + y - a = 0$ is equal to the product of its distances from the axes.

**50. Circle in Polar Coordinates.** Let us now express the *equation of a circle in polar coordinates.* If $C(r_1, \phi_1)$ is the center of a circle of radius $a$ (Fig. 32) and $P(r, \phi)$ any point of the circle, then by the cosine law of trigonometry

$$r^2 + r_1^2 - 2r_1r\cos(\phi - \phi_1) = a^2.$$

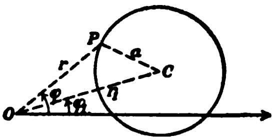

FIG. 32

This is the equation of the circle since, for given values of $r_1$, $\phi_1$, $a$, it is satisfied by the coordinates $r$, $\phi$ of every point of the circle, and by the coordinates of no other point.

Two special cases are important:

(1) If the origin $O$ be taken on the circumference and the polar axis along a diameter $OA$ (Fig. 33), the equation becomes

$$r^2 + a^2 - 2ar\cos\phi = a^2,$$

*i.e.*
$$r = 2a\cos\phi.$$

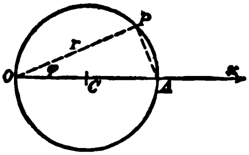

FIG. 33

This equation has a simple geometrical interpretation: the radius vector of any point $P$ on the circle is the projection of the diameter $OA = 2a$ on the direction of the radius vector.

(2) If the origin be taken at the center of the circle, the equation is
$$r = a.$$

### EXERCISES

1. Draw the following circles in polar coordinates:

(*a*) $r = 10 \cos \phi$. (*b*) $r = 2a \cos (\phi - \frac{1}{3}\pi)$. (*c*) $r = \sin \phi$.
(*d*) $r = 6$. (*e*) $r = 7 \sin (\phi - \frac{1}{6}\pi)$. (*f*) $r = 17 \cos \phi$.

2. Write the equation of the circle in polar coordinates:

(*a*) with center at $(10, \frac{1}{2}\pi)$ and radius 5;
(*b*) with center at $(6, \frac{1}{4}\pi)$ and touching the polar axis;
(*c*) with center at $(4, \frac{3}{2}\pi)$ and passing through the origin;
(*d*) with center at $(3, \pi)$ and passing through the point $(4, \frac{1}{3}\pi)$.

3. Change the equations of Ex. (1) and (2) to rectangular coordinates with the origin at the pole and the axis $Ox$ coincident with the polar axis.

4. Determine in polar coordinates the locus of the midpoints of the chords drawn from a fixed point of a circle.

**51. Intersection of Line and Circle.** To solve two equations in $x$ and $y$ of which one is of the first degree (linear) while the other is of the second degree, it is generally most convenient to *solve the linear equation for either $x$ or $y$ and to substitute the value so found in the equation of the second degree. It then remains to solve a quadratic equation.*

The method for solving a quadratic equation consists in completing the square of the terms in $x^2$ and $x$. The equation

$$ax^2 + bx + c = 0$$

has the roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the *discriminant* of the equation. According as the discriminant is positive, zero, or negative, the roots are real and different, real and equal, or imaginary.

An equation of the first degree represents a straight line. If the given equation of the second degree be of the form described in § 47, it will represent a circle. By solving two such simultaneous equations we find the coordinates of the points that lie both on the line and on the circle, *i.e. the points of intersection of line and circle.*

Let us find the intersections of the line

$$y = mx + b$$

with the circle about the origin

$$x^2 + y^2 = r^2.$$

Substituting the value of $y$ from the former equation into the latter, we find the quadratic equation in $x$:

$$x^2 + (mx + b)^2 = r^2,$$

or

$$(1 + m^2)x^2 + 2\,mbx + b^2 - r^2 = 0.$$

The two roots $x_1$, $x_2$ of this equation are the abscissas of the points of intersection; the corresponding ordinates are found by substituting $x_1$, $x_2$ in $y = mx + b$.

It is easily seen that the abscissas $x_1$, $x_2$ are real and different if

$$(1 + m^2)\,r^2 - b^2 > 0,$$

*i.e.* if

$$\frac{b}{\sqrt{1 + m^2}} < r.$$

Since $m = \tan\alpha$, and hence $1/\sqrt{1 + m^2} = \cos\alpha$, the preceding relation means that $b \cos\alpha < r$, *i.e.* the line has a distance from the origin less than the radius of the circle. If

$$(1 + m^2)\,r^2 - b^2 = 0,$$

the roots $x_1$, $x_2$ are real and equal. The line and the circle then have only a single point in common. Such a line is said to ***touch*** the circle or to be a ***tangent*** to the circle. If

$$(1 + m^2)\,r^2 - b^2 < 0,$$

the roots are complex, and the line has no points in common with the circle.

**52. The General Case.** The intersections of the line and circle

$$Ax + By + C = 0,$$
$$x^2 + y^2 + ax + by + c = 0,$$

are found in the same way: substitute the value of $y$ (or $x$), found from the equation of the line, in the equation of the circle and solve the resulting quadratic equation.

It is often desired to determine merely *whether the line is tangent to the circle.* To answer this question, substitute $y$ (or $x$) from the linear equation in the equation of the circle and, *without solving the quadratic equation,* write down the condition for equal roots ($b^2 = 4\,ac$, § 51).

### EXERCISES

1. Find the coordinates of the points where the circle $x^2 + y^2 - x + y - 12 = 0$ crosses the axes.

2. Find the intersections of the line $3\,x + y - 5 = 0$ and the circle $x^2 + y^2 - 22\,x - 4\,y + 25 = 0$.

3. Find the intersections of the line $2\,x - 7\,y + 5 = 0$ and the circle $2\,x^2 + 2\,y^2 + 9\,x + 9\,y - 11 = 0$.

4. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ that are parallel to the line $y = -3\,x + 8$.

5. Show that the equations of the tangents to the circle $x^2 + y^2 = r^2$ with slope $m$ are $y = mx \pm r\sqrt{1 + m^2}$.

6. For what value of $r$ will the line $3\,x - 2\,y - 5 = 0$ be tangent to the circle $x^2 + y^2 = r^2$ ?

7. Find the equations of the tangents to the circle $2\,x^2 + 2\,y^2 - 3\,x + 5y - 7 = 0$ that are perpendicular to the line $x + 2\,y + 3 = 0$.

8. Find the midpoint of the chord intercepted by the line $5x - y + 9 = 0$ on the circle $x^2 + y^2 = 18$.

9. Find the equations of the tangents to the circle $x^2 + y^2 = 58$ that pass through the point (10, 4).

**53. The Tangent to a Circle.** The tangent to a circle (compare § 40) at any point $P$ may be defined as the perpendicular through $P$ to the radius passing through $P$. To find the equation of the tangent to a circle whose center is at the origin,

$$x^2 + y^2 = r^2,$$

at the point $P(x, y)$ of the circle (Fig. 34), observe that the distance $p$ of the tangent from the origin is equal to the radius $r$ and that the angle $\beta$ made by this distance with the axis $Ox$ is such that

$$\cos\beta = \frac{x}{r},\ \sin\beta = \frac{y}{r};$$

substituting these values in the normal form $X \cos\beta + Y \sin\beta = p$ of the equation of a line (§ 40), we find as equation of the tangent

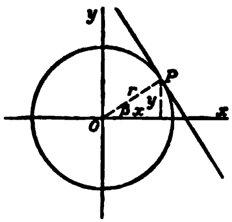

FIG. 34

$$xX + yY = r^2,$$

where $x$, $y$ are the coordinates of the point of contact $P$ and $X$, $Y$ are those of any point of the tangent.

**54. The General Case.** To find the equation of the tangent to a circle whose center is not at the origin let us write the general equation (4), § 48, *viz.*

$$Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0, \tag{4}$$

in the form

$$\left(x + \frac{G}{A}\right)^2 + \left(y + \frac{F}{A}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A},$$

where $-G/A$, $-F/A$ are the coordinates of the center and $G^2/A^2 + F^2/A^2 - C/A$ is the square of the radius $r$ (§ 48). With respect to parallel axes through the center the same circle has the equation

$$x^2 + y^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A} = r^2,$$

and the tangent at the point $P(x, y)$ of the circle is (§ 53):

$$xX + yY = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A} = r^2.$$

Hence, transferring back to the original axes, we find as equation of the tangent at $P(x, y)$ to the circle (4):

$$AxX + AyY + G(x+X) + F(y+Y) + C = 0.$$

This general form of the tangent is readily remembered if we observe that it can be derived from the equation (4) of the circle by replacing $x^2$ by $xX$, $y^2$ by $yY$, $2x$ by $x+X$, $2y$ by $y+Y$.

## EXERCISES

**1.** Find the tangent to the given circle at the given point:

(*a*) $x^2 + y^2 = 41$, $(5, -4)$.
(*b*) $x^2 + y^2 + 6x + 5y - 16 = 0$, $(-2, 3)$.
(*c*) $3x^2 + 3y^2 + 10x + 17y + 18 = 0$, $(-2, -5)$.
(*d*) $x^2 + y^2 - ax - by = 0$, $(a, b)$.

**2.** The equation of any circle through the origin can be written in the form (§ 49) $x^2 + y^2 + ax + by = 0$; show that the line $ax + by = 0$ is the tangent at the origin, and find the equation of the parallel tangent.

**3.** Derive the equation of the tangent to the circle $(x-h)^2 + (y-k)^2 = r^2$.

**4.** Show that the circles $x^2 + y^2 - 6x + 2y + 2 = 0$ and $x^2 + y^2 - 4y + 2 = 0$ touch at the point $(1, 1)$.

**5.** Find the tangents to the circle $x^2 + y^2 - 2x - 10y + 9 = 0$ at the extremities of the diameter through the point $(-1, 11/2)$.

**6.** The line $2x + y = 10$ is tangent to the circle $x^2 + y^2 = 20$; what is the point of contact?

**7.** What is the point of contact if $Ax + By + C = 0$ is tangent to the circle $x^2 + y^2 = r^2$?

**8.** Show that $x - y - 1 = 0$ is tangent to the circle $x^2 + y^2 + 4x - 10y - 3 = 0$, and find the point of contact.

**9.** By § 51, the line $y = mx + b$ has but one point in common with the circle $x^2 + y^2 = r^2$ if $(1 + m^2)r^2 = b^2$; show that in this case the radius drawn to the common point is perpendicular to the line $y = mx + b$.

**55. Circle through Three Points.** *To find the equation of the circle passing through three points* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, observe that the coordinates of these points satisfy the equation of the circle (§ 49)

$$x^2 + y^2 + ax + by + c = 0; \tag{6}$$

hence we must have

$$\begin{cases} x_1^2 + y_1^2 + ax_1 + by_1 + c = 0, \\ x_2^2 + y_2^2 + ax_2 + by_2 + c = 0, \\ x_3^2 + y_3^2 + ax_3 + by_3 + c = 0. \end{cases} \tag{7}$$

From the last three equations we can find the values of $a$, $b$, and $c$; these values must then be substituted in the first equation.

In general this is a long and tedious operation. What we actually wish to do is to eliminate $a$, $b$, $c$ between the *four* equations above. The theory of determinants furnishes a very simple means of eliminating four quantities between four *homogeneous* linear equations. Our equations are *not* homogeneous in $a$, $b$, $c$. But if we write the first two terms in each equation with the factor 1: $(x^2 + y^2) \cdot 1$, $(x_1^2 + y_1^2) \cdot 1$, etc., we have four equations which are linear and homogeneous in 1, $a$, $b$, $c$; hence the result of eliminating these four quantities is the determinant of their coefficients equated to zero. Thus the *equation of the circle through three points* is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Compare § 34, where the equation of the straight line through two points is given in determinant form.

### EXERCISES

**1.** Find the equations of the circles that pass through the points:

(*a*) $(2, 3)$, $(-1, 2)$, $(0, -3)$.

(*b*) $(0, 0)$, $(1, -4)$, $(5, 0)$.

(*c*) $(0, 0)$, $(a, 0)$, $(0, b)$.

**2.** Find the circles through the points $(3, -1)$, $(-1, -2)$ which touch the axis $Ox$.

**3.** Find the circle through the points $(2, 1)$, $(-1, 3)$ with center on the line $3x - y + 2 = 0$.

**4.** Find the circle whose center is $(3, -2)$ and which touches the line $3x + 4y - 12 = 0$.

**5.** Find the circle through the origin that touches the line

$$4x - 5y - 14 = 0 \text{ at } (6, 2).$$

**6.** Find the circle inscribed in the triangle determined by the lines

$$24x - 7y + 3 = 0,\ 3x - 4y - 9 = 0,\ 5x + 12y - 50 = 0.$$

**7.** Two circles are said to be *orthogonal* if their tangents at a point of intersection are perpendicular; the square of the distance between their centers is then equal to the sum of the squares of their radii. If the equations of two intersecting circles are

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0, \text{ and } x^2 + y^2 + a_2x + b_2y + c_2 = 0,$$

show that the circles are orthogonal when $a_1a_2 + b_1b_2 = 2(c_1 + c_2)$.

**8.** Find the circle that has its center at $(-2, 1)$ and is orthogonal to the circle $x^2 + y^2 - 6x + 3 = 0$.

**9.** Find the circle that has its center on the line $y = 3x + 4$, passes through the point $(4, -3)$, and is orthogonal to the circle

$$x^2 + y^2 + 13x + 5y + 2 = 0.$$

**56. Inversion.** A circle of center $O$ and radius $a$ being given (Fig. 35), we can find to every point $P$ of the plane (excepting the center $O$) one and only one point $P'$ on $OP$, produced beyond $P$ if necessary, such that

$$OP \cdot OP' = a^2.$$

The point $P'$ is said to be *inverse* to $P$ with respect to the circle $(O, a)$; and as the relation is not

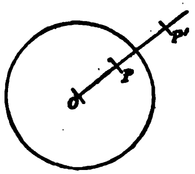

FIG. 35

changed by interchanging $P$ and $P'$, the point $P$ is inverse to $P'$. The point $O$ is called the *center of inversion.*

It is clear that (1) the inverse of a point $P$ within the circle is a point $P'$ without, and *vice versa;* (2) the inverse of a point of the circle itself coincides with it; (3) as $P$ approaches the center $O$, its inverse $P'$ moves off to infinity, and *vice versa.*

The *inverse of any geometrical figure* (line, curve, area, etc.) is the figure formed by the points inverse to all the points of the given figure.

### 57. Inverse of a Circle.

Taking rectangular axes through $O$ (Fig. 36), we find for the relations between the coordinates of two inverse points $P(x, y)$, $P'(x', y')$, if we put $OP = r$, $OP' = r'$:

$$\frac{x'}{x} = \frac{y'}{y} = \frac{r'}{r} = \frac{rr'}{r^2} = \frac{a^2}{r^2}$$

since $rr' = a^2$; hence

$$x' = \frac{a^2x}{x^2 + y^2}, \quad y' = \frac{a^2y}{x^2 + y^2};$$

and similarly

$$x = \frac{a^2x'}{x'^2 + y'^2}, \quad y = \frac{a^2y'}{x'^2 + y'^2}.$$

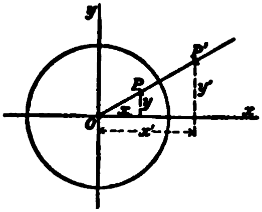

FIG. 36

These equations enable us to find to any curve whose equation is given the equation of the inverse curve, by simply substituting for $x$, $y$ their values.

Thus it can be shown that *by inversion any circle is transformed into a circle or a straight line.*

For, if in the general equation of the circle

$$A(x^2 + y^2) + 2\,Gx + 2\,Fy + C = 0$$

we substitute for $x$ and $y$ the above values, we find

$$Aa^4\frac{x'^2 + y'^2}{(x'^2 + y'^2)^2} + 2\,Ga^2\frac{x'}{x'^2 + y'^2} + 2\,Fa^2\frac{y'}{x'^2 + y'^2} + C = 0,$$

that is, $$Aa^4 + 2\,Ga^2x' + 2\,Fa^2y' + C(x'^2 + y'^2) = 0,$$

which is again the equation of a circle, provided $C \neq 0$. In the special case when $C = 0$, the *given* circle passes through the origin, and its inverse is a straight line. Thus *every circle through the origin is transformed by inversion into a straight line.* It is readily proved conversely that every straight line is transformed into a circle passing through the origin; and in particular that every line through the origin is transformed into itself, as is obvious otherwise.

## EXERCISES

**1.** Find the coordinates of the points inverse to (4, 3), (2, 0), (– 5, 1) with respect to the circle $x^2 + y^2 = 25$.

**2.** Show that by inversion every line (except a line through the center) is transformed into a circle passing through the center of inversion.

**3.** Show that all circles with center at the center of inversion are transformed by inversion into concentric circles.

**4.** Find the equation of the circle about the center of inversion which is transformed into itself.

**5.** With respect to the circle $x^2 + y^2 = 16$, find the equations of the curves inverse to:

(*a*) $x=5$, (*b*) $x-y=0$, (*c*) $x^2+y^2-6x=0$, (*d*) $x^2+y^2-10y+1=0$, (*e*) $3x-4y+6=0$.

**6.** Show that the circle $Ax^2 + Ay^2 + 2Gx + 2Fy + a^2A = 0$ is transformed into itself by inversion with respect to the circle $x^2 + y^2 = a^2$.

**7.** Prove the statements at the end of § 57.

**58. Pole and Polar.** Let $P$, $P'$ (Fig. 37) be *inverse* points with respect to the circle $(O, a)$; then the perpendicular $l$ to $OP$ through $P'$ is called the ***polar*** of $P$, and $P$ the ***pole*** of the line $l$, with respect to the circle.

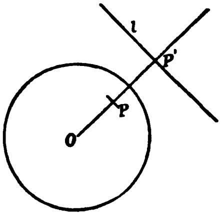

FIG. 37

Notice that (1) if (as in Fig. 37) $P$ lies within the circle, its polar $l$ lies outside; (2) if $P$ lies outside the circle, its polar intersects the circle in two points; (3) if $P$ lies on the circle, its polar is the tangent to the circle at $P$.

Referring the circle to rectangular axes through its center (Fig. 38) so that its equation is

$$x^2 + y^2 = a^2,$$

we can find the equation of the polar $l$ of any given point $P(x, y)$. For, using as equation of the polar the normal form $X \cos \beta + Y \sin \beta = p$, we have evidently, if $P'$ is the point inverse to $P$:

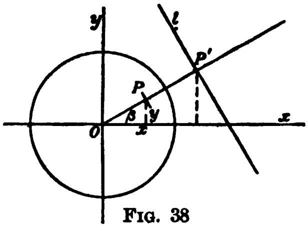

FIG. 38

$$\cos\beta = \frac{x}{\sqrt{x^2+y^2}},\ \sin\beta = \frac{y}{\sqrt{x^2+y^2}},\ p = OP' = \frac{a^2}{\sqrt{x^2+y^2}};$$

therefore the equation becomes

$$\frac{xX}{\sqrt{x^2+y^2}} + \frac{yY}{\sqrt{x^2+y^2}} = \frac{a^2}{\sqrt{x^2+y^2}},$$

or simply

$$xX + yY = a^2.$$

This then is *the equation of the polar $l$ of the point $P(x, y)$ with respect to the circle of radius $a$ about the origin.* If, in particular, the point $P(x, y)$ lies on the circle, the same equation represents the *tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x, y)$*, as shown previously in § 53.

**59. Chord of Contact.** *The polar $l$ of any outside point $P$ with respect to a given circle passes through the points of contact $C_1$, $C_2$ of the tangents drawn from $P$ to the circle.*

To prove this we have only to show that if $C_1$ is one of the points of intersection of the polar $l$ of $P$ with the circle, then the angle $OC_1P$ (Fig. 39) is a right angle. Now the triangles $OC_1P$ and $OP'C_1$ are similar since they have the angle at $O$ in common and the including sides proportional owing to the relation

$$OP \cdot OP' = a^2,$$

*i.e.*

$$\frac{OP}{a} = \frac{a}{OP'},$$

where $a = OC_1$. It follows that $\measuredangle\, OC_1P = OP'C_1 = \frac{1}{2}\pi$.

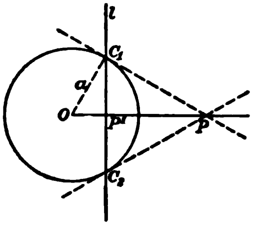

FIG. 39

The rectilinear segment $C_1C_2$ is sometimes called the *chord of contact* of the point $P$. We have therefore proved that *the chord of contact of any outside point $P$ lies on the polar of $P$.*

It follows that *the equations of the tangents that can be drawn from any outside point $P$ to a given circle* can be found by determining the intersections $C_1$, $C_2$ of the polar of $P$ with the circle; the tangents are then obtained as the lines joining $C_1$, $C_2$ to $P$.

**60. The General Case.** The equation of the polar of a point $P(x, y)$ with respect to any circle given in the general form (4), § 48, viz.,

$$(4) \qquad Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,$$

is found by the same method that was used in § 54 to generalize the equation of the tangent. Thus, with respect to parallel axes through the center the equation of the circle is

$$x^2 + y^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A};$$

the equation of the polar of $P(x, y)$ with respect to these axes is by § 58:

$$xX + yY = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}.$$

Hence, transferring back to the original axes, we find as *equation of the polar of* $P(x, y)$ *with respect to the circle* (4):

$$AxX + AyY + G(x + X) + F(y + Y) + C = 0.$$

If, in particular, the point $P(x, y)$ lies outside the circle, this polar contains the chord of contact of $P$; if $P$ lies on the circle, the polar becomes the tangent at $P$ (§ 54).

**61. Construction of Polars.** *If a point $P_1$ describes a line $l$, its polar $l_1$ with respect to a given circle $(O, a)$ turns about a fixed point,* viz., the pole $P$ of the line $l$ (Fig. 40). Conversely, *if a line $l_1$ turns about one of its points $P$, its pole $P_1$ with respect to a given circle $(O, a)$ describes a line $l$, viz. the polar of the point $P$.*

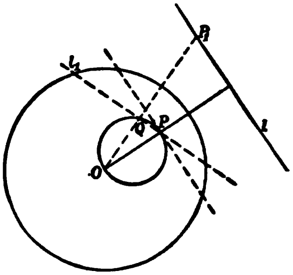

FIG. 40

For, the line $l$ is transformed by inversion with respect to the circle $(O, a)$ into a circle passing through $O$ and through the pole $P$ of $l$; as this circle must obviously be symmetric with respect to $OP$ it must have $OP$ as diameter. Any point $P_1$ of $l$ is transformed by inversion into that point $Q$ of the circle of diameter $OP$ at which this circle is intersected by $OP_1$. The polar of $P_1$ is the perpendicular through $Q$ to $OP_1$; it passes therefore through $P$, wherever $P_1$ be taken on $l$.

The proof of the converse theorem is similar.

The pole $P_1$ of any line $l_1$ can therefore be constructed as the intersection of the polars of any two points of $l_1$; this is of advantage when the line $l_1$ does not meet the circle. And the polar $l_1$ of any point $P_1$ can be constructed as the line joining the poles of any two lines through $P_1$; this is of advantage when the point $P_1$ lies inside the circle.

### EXERCISES

**1.** Find the equation of the polar of the given point with respect to the given circle and sketch if possible:

(*a*) $(4, 7)$, $x^2 + y^2 = 8$.
(*b*) $(0, 0)$, $x^2 + y^2 - 3x - 4 = 0$.
(*c*) $(2, 1)$, $x^2 + y^2 - 4x - 2y + 1 = 0$.
(*d*) $(2, -3)$, $x^2 + y^2 + 3x + 10y + 2 = 0$.

**2.** Find the pole of the given line with respect to the given circle and sketch if possible:

(*a*) $x + 2y - 20 = 0$, $x^2 + y^2 = 20$.
(*b*) $x + y + 1 = 0$, $x^2 + y^2 = 4$.
(*c*) $4x - y = 19$, $x^2 + y^2 = 25$.
(*d*) $Ax + By + C = 0$, $x^2 + y^2 = r^2$.
(*e*) $y = mx + b$, $x^2 + y^2 = r^2$.

**3.** Find the pole of the line joining the points $(20, 0)$ and $(0, 10)$, with respect to the circle $x^2 + y^2 = 25$.

**4.** Find the tangent to the circle $x^2+y^2-10x+4y+9=0$ at $(7, -6)$.

**5.** Find the intersection of the tangents to the circle $2x^2 + 2y^2 - 15x + y - 28 = 0$ at the points $(3, 5)$ and $(0, -4)$.

**6.** Find the tangents to the circle $x^2 + y^2 - 6x - 10y + 2 = 0$ that pass through the point $(3, -3)$.

**7.** Find the tangents to the circle $x^2 + y^2 - 3x + y - 10 = 0$ that pass through the point $(-\frac{8}{3}, -\frac{14}{3})$.

**8.** Show that the distances of two points from the center of a circle are proportional to the distances of each from the polar of the other.

**9.** Show analytically that if two points are given such that the polar of one point passes through the second point, then the polar of the second point passes through the first point.

**10.** Find the poles of the lines $x - y - 3 = 0$ and $x + y + 8 = 0$ with respect to the circle $x^2 + y^2 - 6x + 4y + 3 = 0$.

### 62. Power of a Point.

If in the left-hand member of the equation of the circle

$$(x-h)^2+(y-k)^2-r^2=0,$$

we substitute for $x$ and $y$ the coordinates $x_1$, $y_1$ of a point $P_1$ not on the circle (Fig. 41), the expression $(x_1-h)^2+(y_1-k)^2-r^2$ is different from zero. Its value is called *the power of the point $P_1$ $(x_1, y_1)$ with respect to the circle.* As $(x_1-h)^2+(y_1-k)^2$ is the square of the distance $P_1C = d$ between the point $P_1$ $(x_1, y_1)$ and the center $C(h, k)$, the power of the point $P_1$ $(x_1, y_1)$ with respect to the circle is $d^2-r^2$; and this is positive for points without the circle $(d > r)$, zero for points on the circle $(d = r)$, and negative for points within the circle $(d < r)$.

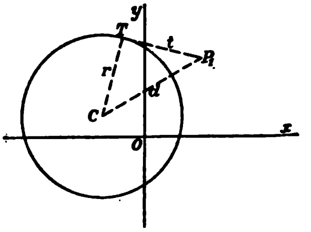

FIG. 41

If the point lies without the circle, its power has a simple interpretation; it is the square of the segment $P_1T = t$ of the tangent drawn from $P_1$ to the circle:

$$t^2 = d^2 - r^2 = (x_1-h)^2+(y_1-k)^2-r^2.$$

Hence the length $t$ of the tangent that can be drawn from an outside point $P_1$ $(x_1, y_1)$ to a circle $x^2+y^2+ax+by+c=0$ is given by

$$t^2 = x_1^2+y_1^2+ax_1+by_1+c.$$

Notice that the coefficients of $x^2$ and $y^2$ must be 1. Compare the similar case of the distance of a point from a line (§ 42).

### 63. Radical Axis.

The locus of a point whose powers with respect to any two circles

$$x^2+y^2+a_1x+b_1y+c_1=0,$$
$$x^2+y^2+a_2x+b_2y+c_2=0,$$

are equal is given by the equation

$$x^2+y^2+a_1x+b_1y+c_1=x^2+y^2+a_2x+b_2y+c_2,$$

which reduces to

$$(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)=0.$$

This locus is therefore a straight line; it is called the ***radical axis*** of the two circles. It always exists unless $a_1 = a_2$ and $b_1 = b_2$, *i.e.* unless the circles are concentric.

Three circles taken in pairs have three radical axes which pass through a common point, called the ***radical center***. For, if the equation of the third circle is

$$x^2 + y^2 + a_3x + b_3y + c_3 = 0,$$

the equations of the radical axes will be

$$(a_2 - a_3)x + (b_2 - b_3)y + (c_2 - c_3) = 0,$$
$$(a_3 - a_1)x + (b_3 - b_1)y + (c_3 - c_1) = 0,$$
$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) = 0.$$

These lines intersect in a point, since the determinant of the coefficients in these equations is equal to zero (Ex. 3, p. 38).

### 64. Family of Circles.

The equation

$$(8) \qquad (x^2 + y^2 + a_1x + b_1y + c_1) + \kappa(x^2 + y^2 + a_2x + b_2y + c_2) = 0$$

represents *a family, or pencil, of circles each of which passes through the points of intersection of the circles*

$$(9) \qquad x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$

*and*

$$(10) \qquad x^2 + y^2 + a_2x + b_2y + c_2 = 0,$$

if these circles intersect. For, the equation (8) written in the form

$$(1 + \kappa)x^2 + (1 + \kappa)y^2 + (a_1 + \kappa a_2)x + (b_1 + \kappa b_2)y + c_1 + \kappa c_2 = 0$$

represents a circle for every value of $\kappa$ except $\kappa = -1$, as the coefficients of $x^2$ and $y^2$ are equal and there is no $xy$-term (§ 47). Each one of the circles (8) passes through the common points of the circles (9) and (10) if they have any, since the equation (8) is satisfied by the coordinates of those points which satisfy both (9) and (10). Compare § 44. The constant $\kappa$ is called the *parameter* of the family.

In the special case when $\kappa = -1$, the equation is of the first degree and hence represents a line, viz. the radical axis (§ 63) of the two circles (9), (10). If the circles intersect, the radical axis contains their *common chord.*

#### EXERCISES

**1.** Find the powers of the following points with respect to the circle $x^2 + y^2 - 3x - 2y = 0$ and thus determine their positions relative to the circle: (2, 0), (0, 0), (0, −4), (3, 2).

**2.** What is the length of the tangent to the circle: ($a$) $x^2 + y^2 + ax + by + c = 0$ from the point (0, 0), ($b$) $(x - 2)^2 + (y - 3)^2 - 1 = 0$ from the point (4, 4)?

**3.** By § 62, $t^2 = d^2 - r^2 = (d + r)(d - r)$; interpret this relation geometrically.

**4.** Find the radical axis of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ and the length of the common chord.

**5.** Find the radical center of the circles $x^2 + y^2 - 3x + 4y - 7 = 0$, $x^2 + y^2 = 16$, $2(x^2 + y^2) + 6x + 1 = 0$. Sketch the circles and their radical axes.

**6.** Find the circle that passes through the intersections of the circles $x^2 + y^2 + 5x = 0$ and $x^2 + y^2 + x - 2y - 5 = 0$, and (*a*) passes through the point $(-5, 6)$, (*b*) has its center on the line $4x - 2y - 15 = 0$, (*c*) has the radius 5.

**7.** Sketch the family of circles $x^2 + y^2 - 6y + \kappa(x^2 + y^2 + 3y) = 0$.

**8.** What family of circles does the equation $Ax + By + C + \kappa(x^2 + y^2 + ax + by + c) = 0$ represent?

**9.** Find the family of curves inverse to the family of lines $y = mx + b$; (*a*) with $m$ constant and $b$ variable, (*b*) with $m$ variable and $b$ constant. Draw sketches for each case.

**10.** Show that a circle can be drawn orthogonal to three circles, provided their centers are not in a straight line.

**11.** Find the locus of a point whose power with respect to the circle $2x^2 + 2y^2 - 5x + 11y - 6 = 0$ is equal to the square of its distance from the origin. Sketch.

**12.** Find the locus of a point if the sum of the squares of its distances from the sides of an equilateral triangle of side $2a$ is constant.

**13.** Show that the circle through the points $(2, 4)$, $(-1, 2)$, $(3, 0)$ is orthogonal to the circle which is the locus of a point the ratio of whose distances from the points $(2, 3)$ and $(-1, 2)$ is 3. Sketch.

**14.** Show that the circles through two fixed points, say $(-a, 0)$, $(a, 0)$, form a family like that of Ex. 8.

**15.** The locus of a point whose distances from the fixed points $(-a, 0)$, $(a, 0)$ are in the constant ratio $\kappa$ $(\neq 1)$ is the circle

$$x^2 + y^2 + 2\frac{1 + \kappa^2}{1 - \kappa^2}ax + a^2 = 0.$$

Compare Ex. 9, p. 54. Show that, whatever $\kappa (\neq 1)$, this circle intersects every circle of the family of Ex. 15 at right angles.

# CHAPTER V

## POLYNOMIALS

### PART I. QUADRATIC FUNCTION — PARABOLA

**65. Linear Function.** As mentioned in § 28, an expression of the form $mx + b$, where $m$ and $b$ are given real numbers ($m \neq 0$) while $x$ may take any real value, is called a *linear function* of $x$. We have seen that this function is represented graphically by the ordinates of the *straight line*

$$y = mx + b;$$

$b$ is the value of $y$ for $x = 0$, and $m$ is the *slope* of the line, *i.e.* the *rate of change* of the function $y$ with respect to $x$.

**66. Quadratic Function. Parabola.** An expression of the form $ax^2 + bx + c$ in which $a \neq 0$ is called a ***quadratic function*** of $x$, and the curve

$$y = ax^2 + bx + c,$$

whose ordinates represent the function, is called a ***parabola.***

If the *coefficients* $a$, $b$, $c$ are given numerically, any number of points of this curve can be located by arbitrarily assigning to the abscissa $x$ any series of values and computing from the equation the corresponding values of the ordinates. This process is known as *plotting the curve by points;* it is somewhat laborious; but a study of the nature of the quadratic function will show that the determination of a few points is sufficient to give a good idea of the curve.

**67. The Form $y = ax^2$.** Let us first take $b = 0$, $c = 0$; the resulting equation

$$(1) \qquad y = ax^2$$

represents a parabola which passes through the origin, since the values 0, 0 satisfy the equation. *This parabola is symmetric with respect to the axis $Oy$*; for, the values of $y$ corresponding to any two equal and opposite values of $x$ are equal. This line of symmetry is called the ***axis*** of the parabola; its intersection with the parabola is called the ***vertex***.

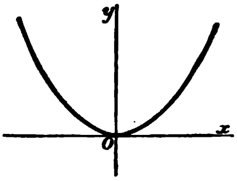

FIG. 42

We may distinguish two cases according as $a > 0$ or $a < 0$; if $a = 0$, the equation becomes $y = 0$, which represents the axis $Ox$.

(1) If $a > 0$, the curve lies above the axis $Ox$. For, no matter what positive or negative value is assigned to $x$, $y$ is positive. Furthermore, as $x$ is allowed to increase in absolute value, $y$ also increases indefinitely. Hence the parabola lies in the first and second quadrants with its vertex at the origin and *opens upward*, i.e. *is concave upward* (Fig. 42).

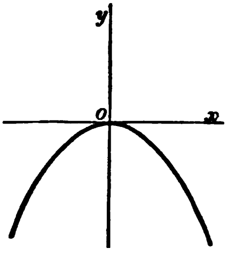

FIG. 43

(2) If $a < 0$, we conclude, similarly, that the parabola lies below the axis $Ox$, in the third and fourth quadrants, with its vertex at the origin and *opens downward*, i.e. *is concave downward* (Fig. 43).

Draw the following parabolas:

$$y = x^2,\ y = 3x^2,\ y = -\tfrac{1}{2}x^2,\ y = \tfrac{1}{4}x^2.$$

**68. The General Equation.** The curve represented by the more general equation

$$(2) \qquad y = ax^2 + bx + c$$

differs from the parabola $y = ax^2$ only in position. To see this we use the process of *completing the square in $x$*; *i.e.* we write the equation in the equivalent form

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c;$$

*i.e.*

$$y - \left(-\frac{b^2}{4a} + c\right) = a\left(x + \frac{b}{2a}\right)^2.$$

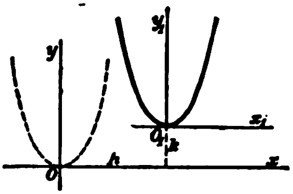

FIG. 44

If we put

$$h = -\frac{b}{2a},\quad k = -\frac{b^2}{4a} + c,$$

the equation becomes

$$y - k = a(x - h)^2,$$

and it is clear (§ 13) that, with reference to parallel axes $O_1x_1$, $O_1y_1$ through the point $O_1$ $(h, k)$ the equation of the curve is $y_1 = ax_1^2$ (Fig. 44). The parabola (2) has therefore the same shape as the parabola $y = ax^2$; but its vertex lies at the point $(h, k)$, and its axis is the line $x = h$. The curve opens upward or downward according as $a > 0$ or $a < 0$.

**69. Nature of the Curve.** To sketch the parabola (2) roughly, it is often sufficient to find the vertex (by completing the square in $x$, as in § 68, and the intersections with the axes. The intercept on the axis $Oy$ is obviously equal to $c$. The intercepts on the axis $Ox$ are found by solving the quadratic equation

$$ax^2 + bx + c = 0.$$

We have thus an interesting interpretation of the roots of any quadratic equation: the roots of $ax^2 + bx + c = 0$ are the abscissas of the points at which the parabola (2) intersects the axis $Ox$. The ordinate of the vertex of the parabola is evidently the least or greatest value of the function $y = ax^2 + bx + c$ according as $a$ is greater or less than zero.

### EXERCISES

**1.** With respect to the same coordinate axes draw the curves $y = ax^2$ for $a=2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2$. What happens to the parabola $y = ax^2$ as $a$ changes ?

**2.** Determine in each of the following examples the value of $a$ so that the parabola $y = ax^2$ will pass through the given point :

(*a*) $(2, 3)$. (*b*) $(-4, 1)$. (*c*) $(-2, -2)$. (*d*) $(3, -4)$.

**3.** A body thrown vertically upward in a vacuum with a velocity of $v$ feet per second will just reach a height of $h$ feet such that $h = \frac{1}{64} v^2$. Draw the curve whose ordinates represent the height as a function of the initial velocity.

(*a*) With what velocity must a ball be thrown vertically upward to rise to a height of 100 ft. ?

(*b*) How high will a bullet rise if shot vertically upward with an initial velocity of 800 ft. per sec., the resistance of the air being neglected ?

**4.** The period of a pendulum of length $l$ (*i.e.* the time of a small back and forth swing) is $T = 2\pi\sqrt{l/g}$. Take $g = 32$ ft./sec. and draw the curve whose ordinates represent the length $l$ of the pendulum as a function of the period $T$.

(*a*) How long is a pendulum that beats seconds (*i.e.* of period 2 sec.) ?

(*b*) How long is a pendulum that makes one swing in two seconds ?

(*c*) Find the period of a pendulum of length one yard.

**5.** Draw the following parabolas and find their vertices and axes :

(*a*) $y = \frac{1}{4}x^2 - x + 2$. (*b*) $y = -\frac{1}{4}x^2 + x$. (*c*) $y = 5x^2 + 15x + 3$.
(*d*) $y = 2 - x - x^2$. (*e*) $y = x^2 - 9$. (*f*) $y = -9 - x^2$.
(*g*) $y = 3x^2 - 6x + 5$. (*h*) $y = \frac{1}{2}x^2 + 2x - 6$. (*i*) $x^2 - 2x - y = 0$.

**6.** What is the value of $b$ if the parabola $y = x^2 + bx - 6$ passes through the point $(1, 5)$ ? of $c$ if the parabola $y = x^2 - 6x + c$ passes through the same point ?

**7.** Suppose the parabola $y = ax^2$ drawn ; how would you draw $y = a(x+2)^2$ ? $y = a(x-7)^2$ ? $y = ax^2 + 2$ ? $y = ax^2 - 7$ ? $y = ax^2 + 2x + 3$ ?

**8.** What happens to the parabola $y = ax^2 + bx + c$ as $c$ changes ? For example, take the parabola $y = x^2 - x + c$, where $c = -3, -2, -1, 0, 1, 2, 3$.

**9.** What happens to the parabola $y = ax^2 + bx + c$ as $a$ changes? For example, take $y = ax^2 - x - 6$, where $a = 2, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -2$.

**10.** (*a*) If the parabola $y = ax^2 + bx$ is to pass through the points $(1, 4)$, $(-2, 1)$ what must be the values of $a$ and $b$? (*b*) Determine the parabola $y = ax^2 + bx + c$ so as to pass through the points $(1, \frac{1}{2})$, $(3, 2)$, $(4, \frac{3}{2})$; sketch.

**11.** The path of a projectile in a vacuum is a parabola with vertical axis, opening downward. With the starting point of the projectile as origin and the axis $Ox$ horizontal, the equation of the path must be of the form $y = ax^2 + bx$. If the projectile is observed to pass through the points $(30, 20)$ and $(50, 30)$, what is the equation of the path? What is the highest point reached? Where will the projectile reach the ground?

**12.** Find the equations of the parabolas determined by the following conditions:

(*a*) the axis coincides with $Oy$, the vertex is at the origin, and the curve passes through the point $(-2, -3)$;

(*b*) the axis is the line $x = 3$, the vertex is at $(3, -2)$, and the curve passes through the origin;

(*c*) the axis is the line $x = -4$, the vertex is $(-4, 6)$, and the curve passes through the point $(1, 2)$.

**13.** Sketch the following parabolas and lines and find the coordinates of their points of intersection:

(*a*) $y = 6x^2$, $y = 7x + 3$.
(*b*) $y = 2x^2 - 3x$, $y = x + 6$.
(*c*) $y = 2 - 3x^2$, $y = 2x + 3$.
(*d*) $y = 3 + x - x^2$, $x + y - 4 = 0$.

**14.** Sketch the following curves and find their intersections:

(*a*) $x^2 + y^2 = 25$, $y = \frac{4}{3}x^2$.
(*b*) $x^2 + y^2 - 6y = 0$, $y = \frac{1}{3}x^2 - 2x + 6$.

**15.** The ordinate of every point of the line $y = \frac{2}{3}x + 4$ is the sum of the corresponding ordinates of the lines $y = \frac{2}{3}x$ and $y = 4$. Draw the last two lines and from them construct the first line.

**16.** The ordinate of every point of the parabola $y = \frac{1}{2}x^2 + \frac{1}{2}x - 1$ is the sum of the corresponding ordinates of the parabola $y = \frac{1}{2}x^2$ and the line $y = \frac{1}{2}x - 1$. From this fact draw the former parabola.

**17.** The ordinate of every point of the parabola $y = \frac{1}{3}x^2 - x + 3$ is the difference of the corresponding ordinates of the parabola $y = \frac{1}{3}x^2$ and the line $y = x - 3$. In this way sketch the former parabola.

**70. Symmetry.** Two points $P_1$, $P_2$ are said to be situated *symmetrically* with respect to a line $l$, if $l$ is the perpendicular bisector of $P_1P_2$; this is also expressed by saying that either point is the *reflection* of the other in the line $l$.

Any two plane figures are said to be symmetric with respect to a line $l$ in their plane if either figure is formed of the reflections in $l$ of all the points of the other figure. Each figure is then the reflection of the other in the line $l$. Two such figures are evidently brought to coincidence by turning either figure about the line $l$ through two right angles. Thus, the lines $y=2x+3$ and $y=-2x-3$ are symmetric with respect to the axis $Ox$.

A line $l$ is called an *axis of symmetry* (or simply an *axis*) of a figure if the portion of the figure on one side of $l$ is the reflection in $l$ of the portion on the other side. Thus, any diameter of a circle is an axis of symmetry of the circle. What are the axes of symmetry of a square? of a rectangle? of a parallelogram?

In analytic geometry, symmetry with respect to the axes of coordinates, and to the lines $y=\pm x$, is of particular importance.

It is readily seen that if a figure is symmetric with respect to *both* axes of coordinates, it is *symmetric with respect to the origin*, *i.e.* to every point $P_1$ of the figure there exists another point $P_2$ of the figure such that the origin bisects $P_1P_2$. A point of symmetry of a figure is also called ***center*** of the figure.

### EXERCISES

1. Give the coordinates of the reflection of the point $(a, b)$ in the axis $Ox$; in the axis $Oy$; in the line $y=x$; in the line $y=2x$; in the line $y=-x$.

2. Show that when $x$ is replaced by $-x$ in the equation of a given curve, we obtain the equation of the reflection of the given curve in the $y$-axis.

**3.** Show that when $x$ and $y$ are replaced by $y$ and $x$, respectively, in the equation of a given curve, we obtain the equation of the reflection of the given curve in the line $y = x$.

**4.** Sketch the lines $y = -2x + 5$ and $x = -2y + 5$ and find their point of intersection.

**5.** Sketch the parabolas $y = x^2$ and $x = y^2$ and find their points of intersection.

**6.** Find the equation of the reflection of the line $2x - 3y + 4 = 0$ in the line $y = x$; in the axis $Ox$; in the axis $Oy$; in the line $y = -x$.

**7.** What is the reflection of the line $x = a$ in the line $y = x$? in the axes?

**8.** Find and sketch the circle which is the reflection of the circle $x^2 + y^2 - 3x - 2 = 0$ in the line $y = x$, and find the points in which the two circles intersect.

**9.** Find the circle which is the reflection of the circle $x^2 + y^2 - 4x + 3 = 0$ in the line $y = x$; in the coordinate axes. Sketch all of these circles.

**10.** What is the general equation of a circle which is its own reflection in the line $y = x$? in the axis $Ox$? in the axis $Oy$? What circle is its own reflection in all three of these lines?

**11.** What is the equation of the reflection of the parabola $y = -x^2 + 4$ in the line $y = x$? in the line $y = -x$? Are these reflections parabolas?

**12.** What is the reflection of the parabola $y = 3x^2 - 5x + 6$ in the axis $Ox$? in the axis $Oy$? Are these reflections parabolas?

**13.** If the cartesian equation of a curve is not changed when $x$ is replaced by $-x$, the curve is symmetric with respect to $Oy$; if it is not changed when $y$ is replaced by $-y$, the curve is symmetric with respect to $Ox$; if it is not changed when $x$ and $y$ are replaced by $-x$ and $-y$, respectively, the curve is symmetric with respect to the origin; if it is not changed when $x$ and $y$ are interchanged, the curve is symmetric with respect to $y = x$.

**71. Slope of Secant.** Let $P(x, y)$ be any point of the parabola

(1) $y = ax^2.$

If $P_1$ $(x_1, y_1)$ be any other point of this parabola so that

(2) $y_1 = ax_1^2,$

the line $PP_1$ (Fig. 45) is called a *secant.*

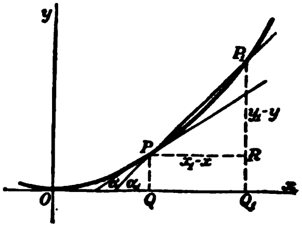

FIG. 45

For the *slope* tan $\alpha_1$ *of this secant* we have from Fig. 45:

$$(3) \qquad \tan \alpha_1 = \frac{RP_1}{QQ_1} = \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x},$$

or, substituting for $y$ and $y_1$ their values:

$$(4) \qquad \tan \alpha_1 = \frac{a(x_1^2 - x^2)}{x_1 - x} = a(x + x_1).$$

**72. Slope of Tangent.** Keeping the point $P$ (Fig. 45) fixed, let the point $P_1$ move along the parabola toward $P$; the limiting position which the secant $PP_1$ assumes at the instant when $P_1$ passes through $P$ is called the ***tangent*** to the parabola at the point $P$.

Let us determine the *slope* tan $\alpha$ *of this tangent.* As the secant turns about $P$ approaching the tangent, the point $Q_1$ approaches the point $Q$, and in the limit $OQ_1 = x_1$ becomes $OQ = x$. The last formula of § 71 gives therefore tan $\alpha$ if we make $x_1 = x$: $\tan \alpha = 2\,ax.$

The slope of the tangent at $P$ which indicates the "steepness" of the curve at $P$ is also called the *slope of the parabola* at $P$. Thus the slope of the parabola $y = ax^2$ at any point whose abscissa is $x$ is $= 2\,ax$; notice that it varies from point to point, being a function of $x$, while the slope of a straight line is constant all along the line.

The knowledge of the slope of a curve is of great assistance in sketching the curve because it enables us, after locating a number of points, to draw the tangent at each point. Thus, for the parabola $y = \frac{2}{3} x^2$ we find $\tan \alpha = \frac{4}{3} x$; locate the points for which $x = 0, 1, 2, -1, -2$, and draw the tangents at these points; then sketch in the curve.

**73. Derivative.** If we think of the ordinate of the parabola $y = ax^2$ as representing the function $ax^2$, the slope of the parabola represents the rate at which the function varies with $x$ and is called the ***derivative*** of the function $ax^2$. We shall denote the derivative of $y$ by $y'$. In § 72 we have proved that the derivative of the function $y = ax^2$ is $y' = 2\,ax$.

The process of finding the derivative of a function, which is called *differentiation*, consists, according to §§ 71–72, in the following steps: Starting with the value $y = ax^2$ of the function for some particular value of $x$ (say, at the point $P$, Fig. 45), we give to $x$ an *increment* $x_1 - x = \Delta x$ (compare § 9) and calculate the value of the corresponding increment $y_1 - y = \Delta y$ of the function. Then *the derivative $y'$ of the function $y$ is the limit that $\Delta y / \Delta x$ approaches as $\Delta x$ approaches zero.* In the case of the function $y = ax^2$ we have

$$\Delta y = y_1 - y = a(x_1^2 - x^2) = a[(x + \Delta x)^2 - x^2] = a[2\,x\Delta x + (\Delta x)^2];$$

hence
$$\frac{\Delta y}{\Delta x} = a(2\,x + \Delta x).$$

The limit of the right-hand member as $\Delta x$ approaches zero gives the derivative:

$$y' = 2\,ax.$$

Thus, the area $y$ of a circle in terms of its radius $x$ is $y = \pi x^2$. Hence the ***derivative*** $y'$, that is the slope of the tangent to the curve that represents the equation $y = \pi x^2$, is $y' = 2\,\pi x$. This represents (§ 72) the *rate of increase* of the area $y$ with respect to $x$. Since $2\,\pi x$ is the length of the circumference, we see that the rate of increase of the area $y$ with respect to the radius $x$ is equal to the circumference of the circle.

**74. Derivative of General Quadratic Function.** By this process we can at once find the derivative of the general quadratic function $y = ax^2 + bx + c$ (§ 66), and hence the slope of the parabola represented by this equation. We have here

$$\begin{aligned}\Delta y &= a(x + \Delta x)^2 + b(x + \Delta x) + c - (ax^2 + bx + c)\\ &= 2\,ax\Delta x + a(\Delta x)^2 + b\Delta x;\end{aligned}$$

hence $\dfrac{\Delta y}{\Delta x} = 2\,ax + b + a\Delta x.$

The limit, as $\Delta x$ becomes zero, is $2\,ax + b$; hence *the derivative of the quadratic function* $y = ax^2 + bx + c$ *is* $y' = 2\,ax + b$.

**75. Maximum or Minimum Value.** It follows both from the definition of the derivative as the limit of $\Delta y/\Delta x$ and from its geometrical interpretation as the slope, $\tan\alpha$, of the curve that *if, for any value of* $x$, *the derivative is positive, the function*, *i.e.* the ordinate of the curve, *is* (*algebraically*) *increasing;* if the derivative is negative, the function is decreasing.

*At a point where the derivative is zero the tangent to the curve is parallel to the axis* $Ox$. The abscissas of the points at which the tangent is parallel to $Ox$ can therefore be found by equating the derivative to zero.

In this way we find that the abscissa of the vertex of the parabola $y = ax^2 + bx + c$ is $x = -b/2\,a$, which agrees with § 68.

We know (§ 68) that the parabola $y = ax^2 + bx + c$ opens upward or downward according as $a$ is $> 0$ or $< 0$. Hence the ordinate of the vertex is a *minimum ordinate, i.e.* algebraically less than the immediately preceding and following ordinates, if $a > 0$; it is a *maximum ordinate, i.e.* algebraically greater than the immediately preceding and following ordinates, if $a < 0$.

This enables us to determine the maximum or minimum of a quadratic function $ax^2 + bx + c$; the value of $x$ for which the function becomes greatest or least is found by equating the derivative to zero; the quadratic function is a maximum or a minimum for this value of $x$ according as $a < 0$ or $> 0$.

Thus, to determine the greatest rectangular area that can be inclosed by a boundary (*e.g.* a fence) of given length $2k$, let one side of the rectangle be called $x$; then the other side is $k - x$. Hence the area $A$ of the rectangle is $A = x(k - x) = kx - x^2$.

Consequently the derivative of $A$ is $k - 2x$. If we set this equal to zero, we have $2x = k$, whence $x = k/2$. It follows that $k - x = k/2$; hence the rectangle of greatest area is a square.

## EXERCISES

**1.** Locate the points of the parabola $y = x^2 - 4x + \frac{5}{2}$ whose abscissas are $-1, 0, 1, 2, 3, 4$, draw the tangents at these points, and then sketch in the curve.

**2.** Sketch the parabolas $4y = -x^2 + 4x$ and $y = x^2 - 3$ by locating the vertex and the intersections with $Ox$ and drawing the tangents at these points.

**3.** Is the function $y = 5(x^2 - 4x + 3)$ increasing or decreasing as $x$ increases from $x = \frac{1}{2}$? from $x = \frac{5}{2}$?

**4.** Find the least or greatest value of the quadratic functions:

(*a*) $2x^2 - 3x + 6$. (*b*) $8 - 6x - x^2$. (*c*) $x^2 - 5x - 5$.
(*d*) $2 - 2x - x^2$. (*e*) $4 + x - \frac{1}{3}x^2$. (*f*) $5x^2 - 20x + 1$.

**5.** Find the derivative of the linear function $y = mx + b$.

**6.** The curve of a railroad track is represented by the equation $y = \frac{1}{3}x^2$, the axes $Ox$, $Oy$ pointing east and north, respectively; in what direction is the train going at the points whose abscissas are $0, 1, 2, -\frac{1}{2}$?

**7.** A projectile describes the parabola $y = \frac{4}{7}x - 3x^2$, the unit being the mile. What is the angle of elevation of the gun? What is the greatest height? Where does the projectile strike the ground?

**8.** A rectangular area is to be inclosed on three sides, the fourth side being bounded by a straight river. If the length of the fence is a constant $k$, what is the maximum area of the rectangle?

## PART II. POLYNOMIALS

**76. The Cubic Function.** A function of the form $a_0x^3 + a_1x^2 + a_2x + a_3$ is called a ***cubic function*** of $x$. The curve represented by the equation

$$y = a_0x^3 + a_1x^2 + a_2x + a_3$$

can be sketched by plotting it by points (§ 66).

For example, to draw the curve represented by the equation

$$y = x^3 - 2x^2 - 5x + 6,$$

we select a number of values of $x$ and compute the corresponding values of $y$:

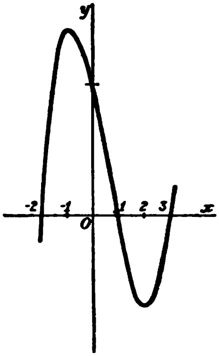

FIG. 46

| $x =$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4 \cdots$ |
|---|---|---|---|---|---|---|---|---|
| $y =$ | $-24$ | $0$ | $8$ | $6$ | $0$ | $-4$ | $0$ | $18 \cdots$ |

These points can then be plotted and connected by a smooth curve which will approximately represent the curve corresponding to the given equation (Fig. 46).

**77. Derivative.** The sketching of such a cubic curve is again greatly facilitated by finding the derivative of the cubic function; the determination of a few points, with their tangents, will suffice to give a good general idea of the curve.

To find the ***derivative*** of the function $y = a_0x^3 + a_1x^2 + a_2x + a_3$ the process of § 73 should be followed. The student may carry this out himself; he will find the quadratic function

$$y' = 3a_0x^2 + 2a_1x + a_2.$$

**78. Maximum or Minimum Values.** The abscissas of those points of the curve at which the tangent is parallel to the axis $Ox$ are again found by equating the derivative to zero; they are therefore the roots of the quadratic equation

$$3\,a_0x^2+2\,a_1x+a_2=0.$$

If at such a point the derivative passes from positive to negative values, the curve is *concave downward*, and the ordinate is a *maximum;* if the derivative passes from negative to positive values, the curve is *concave upward*, and the ordinate is a *minimum.*

**79. Second Derivative.** The derivative of a function of $x$ is in general again a function of $x$. Thus for the cubic function $y=a_0x^3+a_1x^2+a_2x+a_3$ the derivative is the quadratic function

$$y'=3\,a_0x^2+2\,a_1x+a_2.$$

The *derivative of the first derivative* is called the ***second derivative*** of the original function; denoting it by $y''$, we find (§ 74)

$$y''=6\,a_0x+2\,a_1.$$

As a positive derivative indicates an increasing function, while a negative derivative indicates a decreasing function (§ 75), it follows that if at any point of the curve the second derivative is positive, the first derivative, *i.e.* the slope of the curve, increases; geometrically this evidently means that the curve there is *concave upward.* Similarly, if the second derivative is negative, the curve is *concave downward.* We have thus a simple means of telling whether at any particular point the curve is concave upward or downward.

It follows that at any point where the first derivative vanishes, the ordinate is a *minimum* if the second derivative is positive; it is a *maximum* if the second derivative is negative.

**80. Points of Inflexion.** A point at which the curve changes from being concave downward to being concave upward, or *vice versa*, is called a ***point of inflexion.*** At such a point the second derivative vanishes.

Our cubic curve obviously has but one point of inflection, viz. the point whose abscissa is $x=-a_1/(3\,a_0)$.

## EXERCISES

**1.** Find the first and second derivatives of $y$ when:

(*a*) $y = 6x^3 - 7x^2 - x + 2$. (*b*) $y = 20 + 4x - 5x^2 - x^3$.
(*c*) $10y = x^3 - 5x^2 + 3x + 9$. (*d*) $y = (x-1)(x-2)(x-3)$.
(*e*) $y = x^2(x+3)$. (*f*) $7y = 3x - 2x(x^2 - 1)$.

**2.** Sketch the curve $y = (x-2)(x+1)(x+3)$, observing the sign of $y$ between the intersections with $Ox$, and determining the minimum, maximum, and point of inflection.

**3.** In the curve $y = a_0x^3 + a_1x^2 + a_2x + a_3$, what is the meaning of $a_3$?

**4.** Sketch the curves:

(*a*) $5y = (x-1)(x+4)^2$. (*b*) $y = (x-3)^3$.
(*c*) $6y = 6 + x + x^2 - x^3$. (*d*) $y = x^3 - 4x$.
(*e*) $8y = 5x^2 - x^3$. (*f*) $y = x^3 - 3x^2 + 4x - 5$.

**5.** Draw the curves $y = x$, $y = x^2$, $y = x^3$, with their tangents at the points whose abscissas are 1 and $-1$.

**6.** Find the equation of the tangent to the curve $14y = 5x^3 - 2x^2 + x - 20$ at the point whose abscissa is 2.

**7.** At what points of the curve $y = x^3 - 5x^2 + 3$ are the tangents parallel to the line $y = -3x + 5$?

**8.** Are the following curves concave upward or downward at the indicated points? Sketch each of them.

(*a*) $y = 4x^3 - 6x$, at $x = 3$. (*b*) $3y = 5x - 3x^3$, at $x = -2$.
(*c*) $y = x^3 - 2x^2 + 5$, at $x = \frac{1}{3}$. (*d*) $2y = x^3 - 3x^2$, at $x = 1$.
(*e*) $y = 1 - x - x^3$, at $x = 0$. (*f*) $10y = x^3 + x^2 - 15x + 6$, at $x = -\frac{1}{3}$.

**9.** Show that the parabola $y = ax^2 + bx + c$ is concave upward or concave downward for all values of $x$ according as $a$ is positive or negative.

**10.** The angle between two curves at a point of intersection is the angle between their tangents. Find the angles between the curves $y = x^2$ and $y = x^3$ at their points of intersection.

**11.** Find the angle at which the parabola $y = 2x^2 - 3x - 5$ intersects the curve $y = x^3 + 3x - 17$ at the point $(2, -3)$.

**12.** The ordinate of every point of the curve $y = x^3 + 2x^2$ is the sum of the ordinates of the curves $y = x^3$ and $y = 2x^2$. From the latter two curves construct the former.

**13.** From the curve $y = x^3$ construct the following curves:

(*a*) $y = 4x^3$. (*b*) $y = \left(\frac{x}{2}\right)^3$. (*c*) $y = x^3 - 2$. (*d*) $y = 2x^3 + 4$.

**14.** Draw the curve $2y = x^3 - 3x^2$ and its reflection in the line $y = x$. What is the equation of this reflected curve? What is the equation of the reflection in the axis $Oy$?

**15.** A piece of cardboard 18 inches square is used to make a box by cutting equal squares from the four corners and turning up the sides. Draw the curve whose ordinates represent the volume of the box as a function of the side of the square cut out. Find its maximum.

**16.** The strength of a rectangular beam cut from a log one foot in diameter is proportional to (*i.e.* a constant times) the width and the square of the depth. Find the dimensions of the strongest beam which can be cut from the log. Draw the curve whose ordinates represent the strength of the beam as a function of the width.

**17.** Find the equation of the curve in the form $y = ax^3 + bx^2 + cx + d$ which passes through the following points:

(*a*) (0, 0), (2, −1), (−1, 4), (3, 4);
(*b*) (1, 1), (3, −1), (0, 5), (−4, 1).

**18.** Show that every cubic curve of the form $y = a_0x^3 + a_1x^2 + a_2x + a_3$ is symmetric with respect to its point of inflection.

**81. Polynomials.** The methods used in studying the quadratic and cubic functions and the curves represented by them can readily be extended to the general case of the *polynomial*, or *rational integral function*, of the *n*th degree,

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where the coefficients $a_0, a_1, \cdots a_n$ may be any real numbers, while the exponent $n$, which is called the *degree* of the polynomial, is a positive integer.

We shall often denote such a polynomial by the letter $y$ or by the symbol $f(x)$ (read: function of $x$, or $f$ of $x$); its value for any particular value of $x$, say $x = x_1$ or $x = h$, is then denoted by $f(x_1)$ or $f(h)$, respectively. Thus, for $x = 0$ we have $f(0) = a_n$.

**82. Calculation of Values of a Polynomial.** In plotting the curve $y = f(x)$ by points (§§ 66, 76) we have to calculate a number of ordinates. Unless $f(x)$ is a very simple polynomial this is a rather laborious process. To shorten it observe that the value $f(x_1)$ of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

for $x = x_1$ can be written in the form

$$f(x_1) = (\cdots(((a_0x_1 + a_1)x_1 + a_2)x_1 + a_3)x_1 + \cdots + a_{n-1})x_1 + a_n.$$

To calculate this expression begin by finding $a_0x_1 + a_1$; multiply by $x_1$ and add $a_2$; multiply the result by $x_1$ and add $a_3$; etc. This is best carried out in the following form:

$$\begin{array}{llll}
a_0 & a_1 & a_2 & \cdots a_n \\
 & \underline{a_0x_1} & \underline{(a_0x_1 + a_1)x_1} & \\
 & a_0x_1 + a_1 & (a_0x_1 + a_1)x_1 + a_2 \cdots &
\end{array}$$

For instance, if

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 12x + 5 \\ &= ((2x - 3)x - 12)x + 5, \end{aligned}$$

to find $f(3)$ write the coefficients in a row and place $2 \times 3 = 6$ below the second coefficient; the sum is 3. Place $3 \times 3 = 9$ below the third coefficient; the sum is $-3$. Place $3 \times (-3) = -9$ below the last coefficient; the sum, $-4$, is $= f(3)$.

$$\begin{array}{rrrr}
2 & -3 & -12 & 5 \\
 & 6 & 9 & -9 \\
\hline
2 & 3 & -3 & -4
\end{array}$$

This process is useful in calculating the values of $y$ that correspond to various values of $x$, as we have to do in plotting a curve by points.

### EXERCISES

**1.** If $f(x) = 5x^3 - 13x + 2$, what is meant by $f(a)$? by $f(x+h)$? What is the value of $f(0)$? of $f(2)$? of $f(-3/5)$? of $f(-1)$?

**2.** Find the ordinates of the curve $y = x^4 - x^3 + 3x^2 - 12x + 3$ for $x = 3, -9, -\frac{1}{2}$.

**3.** Find the ordinates of $2y = x^4 + 3x^2 - 20x - 25$ for $x = 1, 2, 3, -1, -2$.

**4.** Suppose the curve $y = f(x)$ drawn; how would you sketch:
(*a*) $y = f(x-2)$? (*b*) $y = f(x+3)$? (*c*) $y = f(2x)$? (*d*) $y = f(-x)$? (*e*) $y = f\left(\frac{x}{4}\right)$? (*f*) $y = f(x) + 5$? (*g*) $y = f(x) - 2x$?

**83. Derivative of the Polynomial.** We have seen in the preceding sections how greatly the sketching of a curve and the investigation of a function is facilitated by the use of the derivatives of the function. Thus, in particular, the ***first derivative*** $y'$ is the ***rate of change*** of the function $y$ with $x$, and hence determines the slope, or steepness, of the curve $y = f(x)$. We begin therefore the study of the polynomial by determining its derivative. The method is essentially the same as that used in §§ 73, 74 for finding the derivative of a quadratic function.

The ***first derivative*** $y'$ of any function $y$ of $x$ is defined, as in § 73, to be the *limit of the quotient* $\Delta y/\Delta x$ *as* $\Delta x$ *approaches zero*, $\Delta y$ being the increment of the function $y$ corresponding to the increment $\Delta x$ of $x$; in symbols:

$$y' = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}.$$

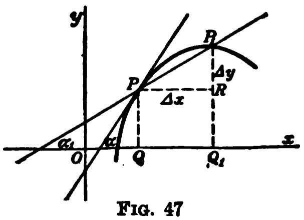

FIG. 47

Geometrically this means that $y'$ is the slope of the tangent of the curve whose ordinate is $y$. For, $\Delta y/\Delta x$ is the *slope of the secant* $PP_1$ (Fig. 47):

$$\frac{\Delta y}{\Delta x} = \tan \alpha_1;$$

and the limit of this quotient as $\Delta x$ approaches zero, *i.e.* as $P_1$ moves along the curve to $P$, is the ***slope of the tangent*** at $P$:

$$y' = \tan \alpha = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}.$$

If the function $y$ be denoted by $f(x)$, then

$$\Delta y = f(x + \Delta x) - f(x);$$

hence

$$y' = \lim_{\Delta x = 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

**84. Calculation of the Derivative.** To find, by means of the last formula, the derivative of the polynomial

$$y = f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

we should have to form first $f(x + \Delta x)$, *i.e.*

$$(x + \Delta x)^n + a_1 (x + \Delta x)^{n-1} + \cdots + a_n,$$

subtract from this the original polynomial, then divide by $\Delta x$, and finally put $\Delta x = 0$.

This rather cumbersome process can be avoided if we observe that a polynomial is a sum of terms of the form $ax^n$ and apply the following fundamental propositions about derivatives:

(1) *the derivative of a sum of terms is the sum of the derivatives of the terms;*

(2) *the derivative of* $ax^n$ *is a times the derivative of* $x^n$*;*

(3) *the derivative of a constant is zero;*

(4) *the derivative of* $x^n$ *is* $nx^{n-1}$.

The first three of these propositions can be regarded as obvious; a fuller discussion of them, based on an exact definition of the limit of a function, is given in the differential calculus. A proof of the fourth proposition is given in the next article.

On the basis of these propositions we find at once that the derivative of the polynomial

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

is

$$y' = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \cdots + a_{n-1}.$$

**85. Derivative of $x^n$.** By the definition of the derivative (§ 83) we have for the derivative of $y = x^n$:

$$y' = \lim_{\Delta x = 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

Now by the binomial theorem we have

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n,$$

and hence

$$(x + \Delta x)^n - x^n = nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$$

Dividing by $\Delta x$ and then letting $\Delta x$ become zero, we find

$$y' = nx^{n-1}.$$

### EXERCISES

**1.** Find the derivatives of the following functions of $x$ by means of the fundamental definition (§ 83) and check by § 84:

(*a*) $x^3$. (*b*) $x^2 + x$. (*c*) $x^4 + 6x^2$.
(*d*) $-6x^3$. (*e*) $x^4 - 3x^3$. (*f*) $mx + b$.

**2.** Find the derivatives of the following functions:

(*a*) $5x^4 - 3x^2 + 6x$. (*b*) $1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$. (*c*) $(x - 2)^3$.
(*d*) $(2x + 3)^5$. (*e*) $3(4x - 1)^3$. (*f*) $x^n + ax^{n-1} + bx^{n-2}$.

**3.** For the following functions write the derivative indicated:

(*a*) $5x^3 - 3x$, find $y'''$. (*b*) $ax^2 + bx + c$, find $y'''$.
(*c*) $x^5$, find $y^{\text{v}}$. (*d*) $ax^3 + bx^2 + cx + d$, find $y^{\text{iv}}$.
(*e*) $\frac{1}{6}x^6$, find $y''$. (*f*) $\frac{1}{6!}x^6$, find $y^{\text{vii}}$.
(*g*) $x^{12} - qx^8$, find $y'''$. (*h*) $(2x - 3)^3$, find $y'''$.

**86. Properties of the General Polynomial Curve.** In plotting the curve

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

observe that (Fig. 48):

(*a*) the intercept $OB$ on the axis $Oy$ is equal to the constant term $a_n$;

(*b*) the intercepts $OA_1$, $OA_2$, $\cdots$ on the axis $Ox$ are roots of the equation $y = 0$, *i.e.*

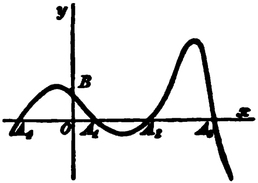

FIG. 48

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0;$$

(*c*) the abscissas of the least and greatest ordinates are found by solving the equation $y' = 0$, *i.e.* (§ 84)

$$na_0x^{n-1} + \cdots + a_{n-1} = 0,$$

every real root giving a minimum ordinate if for this root $y''$ is positive and a maximum ordinate if $y''$ is negative;

(*d*) the abscissas of the points of inflection are found by solving the equation $y'' = 0$, *i.e.*

$$n(n-1)a_0x^{n-2} + \cdots + 2a_{n-2} = 0,$$

every real root of this equation being the abscissa of a point of inflection provided that $y''' \neq 0$. (If $y'''$ were zero, $y'$ might not be a maximum or minimum, and further investigation would be necessary.)

**87. Continuity of Polynomials.** It should also be observed that the function $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is one-valued, real, and finite for every $x$; *i.e.* to every real and finite abscissa $x$ belongs one and only one ordinate, and this ordinate is real and finite. Moreover, as the first derivative $y' = na_0x^{n-1} + \cdots + a_{n-1}$ is again a polynomial, the slope of the curve is everywhere one-valued and finite.

Thus, so-called *discontinuities* of the ordinate (Fig. 49) or of the slope (Fig. 50) cannot occur: the curve $y = a_0x^n + \cdots + a_n$ is *continuous.*

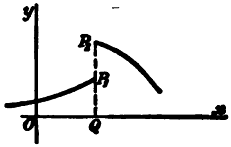

FIG. 49

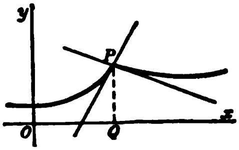

FIG. 50

Strictly defined, the ***continuity*** of the function $y = a_0x^n + \cdots + a_n$ means that, for every value of $x$, *the limit of the function is equal to the value of the function.* The function $y = a_0x^n + \cdots + a_n$ has one and only one value for any value $x = x_1$, viz. $a_0x_1^n + \cdots + a_n$. The value of the function for any other value of $x$, say for $x_1 + \Delta x$, is $a_0(x_1 + \Delta x)^n + \cdots + a_n$ which can be written in the form $a_0x_1^n + \cdots + a_n +$ terms containing $\Delta x$ as factor. Therefore as $\Delta x$ approaches zero, the function approaches a limit, viz. its value for $x = x_1$.

**88. Intermediate Values.** A continuous function, in varying from any value to any other value, must necessarily pass through all intermediate values. Thus, our polynomial $y = a_0x^n + \cdots + a_n$, if it passes from a negative to a positive value (or *vice versa*), must pass through zero. It follows from this that *between any two ordinates of opposite sign the curve $y = a_0x^n + \cdots + a_n$ must cross the axis Ox at least once.*

It also follows from the continuity of the polynomial and its derivatives that *between any two intersections with the axis Ox there must lie at least one maximum or minimum,* and between a maximum and a minimum there must lie a point of inflection.

Ordinates at particular points can be calculated by the process of § 82.

## EXERCISES

**1.** Sketch the following curves:

(*a*) $y=(x-1)(x-2)(x-3)$. (*b*) $4y=x^4-1$. (*c*) $10y=x^5$.
(*d*) $10y=x^5+5$. (*e*) $4y=(x+2)^2(x-3)$. (*f*) $y=(x-1)^4$.

**2.** When is the curve $y=a_0x^n+a_1x^{n-1}+\cdots+a_n$ symmetric with respect to $Oy$?

**3.** Determine the coefficients so that the curve $y=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$ shall touch $Ox$ at $(1, 0)$ and at $(-1, 0)$ and pass through $(0, 1)$, and sketch the curve.

**4.** Find the coordinates of the maxima, minima, and points of inflection and then sketch the curve $4y=x^4-2x^2$.

**5.** Are the following curves concave upward or downward at the indicated points?

(*a*) $16y=16x^4-8x^2+1$, at $x=-1, -\frac{1}{2}, 0, \frac{1}{2}, 3$.
(*b*) $y=4x-x^4$, at $x=-2, 0, 1, 3$.
(*c*) $y=x^n$, at any point; distinguish the cases when $n$ is a positive even or odd integer.

**6.** What happens to the curves $y=ax^3$ and $y=ax^4$ as $a$ changes? For example, take $a=2, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -2$.

**7.** Find the values of $x$ for which the following relations are true:

(*a*) $x^4-6x^2+9 \gtreqless 0$. (*b*) $(x-1)^2(x^2-4) \gtreqless 0$.

**8.** Those curves whose ordinates represent the values of the first, second, etc., derivatives of a given polynomial are called the first, second, etc., *derived curves*. Sketch on the same coordinate axes the following curves and their derived curves:

(*a*) $6y=2x^3-3x^2-12x$. (*b*) $y=(x-2)^2(x+1)$.
(*c*) $y=(x+1)^3$. (*d*) $2y=x^4+x^2+1$.

**9.** At what point on $Ox$ must the origin be taken in order that the equation of the curve $y=2x^3-3x^2-12x-5$ shall have no term in $x^2$? no term in $x$?

**10.** Find, to three significant figures, the roots of the equation

$$x^3-3x+1=0.$$

# CHAPTER VI

## THE PARABOLA

**89. The Parabola.** The ***parabola*** can be defined as the *locus of a point whose distance from a fixed point is equal to its distance from a fixed line.* The fixed point is called the ***focus***, the fixed line the ***directrix***, of the parabola.

Let $F$ (Fig. 51) be the fixed point, $d$ the fixed line; then every point $P$ of the parabola must satisfy the condition

$$FP = PQ,$$

$Q$ being the foot of the perpendicular from $P$ to $d$. Let us take $F$ as origin, or pole, and the perpendicular $FD$ from $F$ to the directrix as polar axis, and let the given distance $FD = 2a$. Then $FP = r$ and $PQ = 2a - r\cos\phi$. The condition $FP = PQ$ becomes therefore

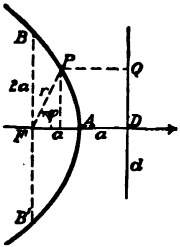

FIG. 51

$$r = 2a - r\cos\phi,$$

*i.e.*

(1) $$r = \frac{2a}{1 + \cos\phi}.$$

This equation, which expresses the radius vector of $P$ as a function of the vectorial angle $\phi$, is *the* ***polar equation*** *of the parabola, when the focus is taken as pole and the perpendicular from the focus to the directrix as polar axis.*

**90. Polar Construction of Parabolas.** By means of the equation (1) the parabola can be plotted by points. Thus, for $\phi = 0$ we find $r = a$ as intercept on the polar axis. As $\phi$ increases from the value 0, $r$ continually increases, reaching the value $2a$ for $\phi = \frac{1}{2}\pi$, and becoming infinite as $\phi$ approaches the value $\pi$.

For any negative value of $\phi$ (between 0 and $-\pi$) the radius vector has the same length as for the corresponding positive value of $\phi$; this means that the parabola is symmetric with respect to the polar axis.

The intersection $A$ of the curve with its axis of symmetry is called the ***vertex***, and the axis of symmetry $FA$ the ***axis***, of the parabola. The segment $BB'$ cut off by the parabola on the perpendicular to the axis drawn through the focus is called the ***latus rectum;*** its length is $4a$, if $2a$ is the distance between focus and directrix. Notice also that the vertex $A$ bisects this distance $FD$ so that the distance between focus and vertex as well as that between vertex and directrix is $a$.

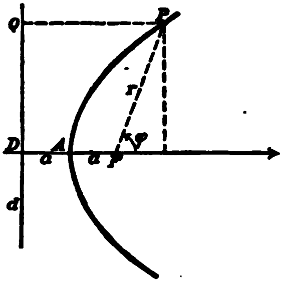

FIG. 52

In Fig. 52 the polar axis is taken positive in the sense from the pole toward the directrix. If the sense from the directrix to the pole is taken as positive (Fig. 52), we have again with $F$ as pole $FP = r$, but the distance of $P$ from the directrix is $2a + r\cos\phi$, so that the polar equation is now

$$(2) \qquad r = \frac{2a}{1 - \cos\phi}.$$

We have assumed $a$ as a positive number, $2a$ denoting the absolute value of the distance between the fixed point (focus) and the fixed line (directrix). The radius vector $r$ is then always positive. But the equations (1) and (2) still represent parabolas if $a$ is a negative number, viz. (1) the parabola of Fig. 52, (2) the parabola of Fig. 51, the radius vector $r$ being negative (§ 16).

**91. Mechanical Construction.** A mechanism for tracing an arc of a parabola consists of a right-angled triangle (shaded in Fig. 53), one of whose sides is applied to the directrix. At a point *R* of the other side *RQ* a string of length *RQ* is attached; the other end of the string is attached at the focus *F*. As the triangle slides along the directrix, the string is kept taut by means of a pencil at *P* which traces the parabola. Of course, only a portion of the parabola can thus be traced, since the curve extends to infinity.

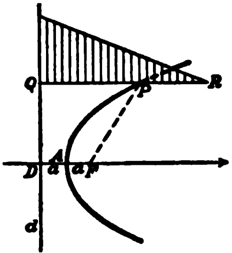

FIG. 53

**92. Transformation to Cartesian Coordinates.** To obtain the cartesian equation of the parabola let the origin *O* be taken at the vertex, *i.e.* midway between the fixed line and fixed point, and the axis *Ox* along the axis of the parabola, positive in the sense from vertex to focus (Fig. 54). Then the focus *F* has the coordinates $a$, 0, and the equation of the directrix is $x = -a$. The distance *FP* of any point $P(x, y)$ of the parabola from the focus is therefore $\sqrt{(x-a)^2 + y^2}$, and the distance *QP* of *P* from the directrix is $a + x$. Hence the equation is

$$(x-a)^2 + y^2 = (a+x)^2,$$

which reduces at once to

$$y^2 = 4ax. \tag{3}$$

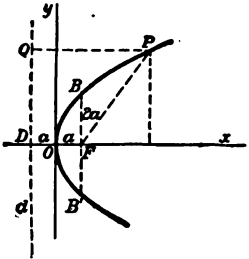

FIG. 54

This then is the *cartesian equation* of the parabola, *referred to vertex and axis, i.e.* when the vertex is taken as origin and the axis of the parabola (from vertex toward focus) as axis *Ox*.

Notice that the ordinate at the focus $(a, 0)$ is of length $2a$; the double ordinate $B'B$ at the focus is the latus rectum (§ 90).

**93. Negative Values of *a*.** In the last article the constant *a* was again regarded as positive; but (compare § 90) the equation (3) still represents a parabola when *a* is a negative number, the only difference being that in this case the parabola turns its opening in the negative sense of the axis $Ox$ (toward the left in Fig. 54). Thus the parabolas $y^2=4ax$ and $y^2=-4ax$ are symmetric to each other with respect to the axis $Oy$ (Ex. 14, p. 77).

The equation (3) is very convenient for plotting a parabola by points. Sketch, with respect to the same axes, the parabolas: $y^2=16x$, $y^2=-16x$, $y^2=x$, $y^2=-x$, $y^2=3x$, $y^2=-\frac{1}{8}x$.

**94. Axis Vertical.** The equation

$$x^2=4ay, \tag{4}$$

which differs from (3) merely by the interchange of $x$ and $y$, evidently represents a parabola whose vertex lies at the origin and whose axis coincides with the axis $Oy$. The parabolas (3) and (4) are each the reflection of the other in the line $y=x$ (Ex. 14, p. 77). The equation (4) can be written in the form

$$y=\frac{1}{4a}x^2.$$

As $1/4a$ may be any constant, this is the equation discussed in § 67.

**95. New Origin.** An equation of the form (Fig. 55)

$$(y-k)^2=4a(x-h), \tag{5}$$

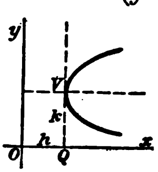

FIG. 55

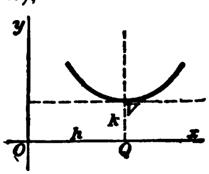

FIG. 56

or of the form (Fig. 56)

$$(x-h)^2=4a(y-k), \tag{6}$$

evidently represents a parabola whose vertex is the point $(h, k)$, while the axis is in the former case parallel to $Ox$, in the latter to $Oy$. For, by taking the point $(h, k)$ as new origin we can reduce these equations to the forms (3), (4), respectively.

The parabola (5) turns its opening to the right or left, the parabola (6) upward or downward, according as $4a$ is positive or negative.

**96. General Equation.** The equations (5), (6) as well as the equations (3), (4) are of the second degree. Now the general equation of the second degree (§ 47),

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

can be reduced to one of the forms (5), (6) if it contains no term in $xy$ and only one of the terms in $x^2$ and $y^2$, *i.e.* if $H = 0$ and either $A$ or $B$ is $= 0$. This reduction is performed (as in § 48) by completing the square in $y$ or $x$ according as the equation contains the term in $y^2$ or $x^2$.

Thus *any equation of the second degree, containing no term in $xy$ and only one of the squares $x^2$, $y^2$, represents a parabola*, whose vertex is found by completing the square and whose axis is parallel to one of the axes of coordinates.

## EXERCISES

1. Sketch the following parabolas:

(*a*) $r = \dfrac{2}{1 + \cos\phi}$. (*b*) $r = \dfrac{10}{1 - \cos\phi}$. (*c*) $r = a \sec^2 \frac{1}{2}\phi$.

2. Sketch the following curves and find their intersections:

(*a*) $r = 8\cos\phi$, $r = \dfrac{2}{1 - \cos\phi}$. (*b*) $r = a$, $r = \dfrac{a}{1 + \cos\phi}$.

(*c*) $r = 4\cos\phi$, $r = \dfrac{8}{1 + \cos\phi}$. (*d*) $r\cos\phi = 2a$, $r = \dfrac{2a}{1 - \cos\phi}$.

3. Sketch the following parabolas:

(*a*) $(y - 2)^2 = 8(x - 5)$. (*b*) $(x + 3)^2 = 5(3 - y)$.

(*c*) $x^2 = 6(y + 1)$. (*d*) $(y + 3)^2 = -3x$.

4. Sketch each of the following parabolas and find the coordinates of the vertex and focus, and the equations of the directrix and axis:

(a) $y^2-2y-3x-2=0$. (b) $x^2+4x-4y=0$.
(c) $x^2-4x+3y+1=0$. (d) $3x^2-6x-y=0$.
(e) $8y^2-16y+x+6=0$. (f) $y^2+y+x=0$.
(g) $x^2-x-3y+4=0$. (h) $8y^2-3x+3=0$.

5. Sketch the following loci and find their intersections:

(a) $y=2x$, $y=x^2$. (b) $y^2=4ax$, $x+y=3a$.
(c) $y^2=x+3$, $y^2=5-x$. (d) $y^2+4x+4=0$, $x^2+y^2=41$.

6. Sketch the parabolas with the following lines and points as directrices and foci, and find their equations:

(a) $x-4=0$, $(6, -2)$. (b) $y+3=0$, $(0, 0)$.
(c) $2x+5=0$, $(0, -1)$. (d) $x=0$, $(2, -3)$.
(e) $3y-1=0$, $(-2, 1)$. (f) $x-2a=0$, $(a, b)$.

7. Find the parabola, with axis parallel to $Ox$, and passing through the points:

(a) $(1, 0)$, $(5, 4)$, $(10, -6)$. (b) $(\frac{10}{3}, -5)$, $(\frac{4}{3}, 0)$, $(\frac{2}{3}, -3)$.

8. Find the parabola, with axis parallel to $Oy$, and passing through the points:

(a) $(0, 0)$, $(-2, 1)$, $(6, 9)$. (d) $(1, 4)$, $(4, -1)$, $(-3, 20)$.

9. Find the parabola whose directrix is the line $3x-4y-10=0$ and whose focus is: (a) at the origin; (b) at $(5, -2)$. Sketch each curve. When does the equation of a parabola contain an $xy$ term?

10. Find the parabolas with the following points as vertices and foci (two solutions):

(a) $(-3, 2)$, $(-3, 5)$. (b) $(2, 5)$, $(-1, 5)$.
(c) $(-1, -1)$, $(1, -1)$. (d) $(0, 0)$, $(0, -a)$.

11. If $s$ denotes the distance (in feet) from a point $P$ in the line of motion of a falling body, at a time $t$ (in seconds),

$$s-s_0=\tfrac{1}{2}g(t-t_0)^2,$$

where $g$ is the gravitational constant (32.2 approximately) and $s_0$ is the distance from $P$ at the time $t_0$, show that this equation can be put in the standard form

$$\bar{s}=\tfrac{1}{2}g\bar{t}^2,$$

where $\bar{s}$ denotes the distance from some other fixed point in the line of motion and $\bar{t}$ is the time since the body was at that point.

**12.** The melting point $t$ (in degrees Centigrade) of an alloy of lead and zinc is found to be

$$t = 133 + .875\,x + .01125\,x^2,$$

where $x$ is the percentage of lead in the alloy. Reduce the equation to standard form $\bar{t} = k\bar{x}^2$; and show that $\bar{x} = x - h$, $\bar{t} = t - k$, where $h$ is the percentage of lead that gives the lowest melting point, and $k$ is the temperature at which that alloy melts.

**13.** Show that the locus of the center of the circle which passes through a fixed point and is tangent to a fixed line is a parabola.

**14.** Show that the locus of the center of a circle which is tangent to a fixed line and a fixed circle is a parabola. Find the directrix of this parabola.

**97. Slope of the Parabola.** The *slope* $\tan\alpha$ of the parabola

$$y^2 = 4\,ax$$

at any point $P(x, y)$ (Fig. 57) can be found (comp. § 72) by first determining the slope

$$\tan\alpha_1 = \frac{y_1 - y}{x_1 - x}$$

of the secant $PP_1$, and then letting $P_1(x_1, y_1)$ move along the curve up to the point $P(x, y)$. Now as $P_1$ comes to coincide with $P$, $x_1$ becomes equal to $x$, and $y_1$ equal to $y$, so that the expression for $\tan\alpha_1$ loses its meaning. But observing that $P$ and $P_1$ lie on the parabola, we have $y^2 = 4\,ax$ and $y_1^2 = 4\,ax_1$, and hence $y_1^2 - y^2 = 4\,a(x_1 - x)$. Substituting from this relation the value of $x_1 - x$ in the above expression for $\tan\alpha_1$, we find for the slope of the secant:

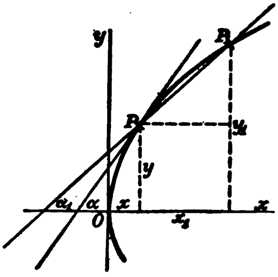

Fig. 57

$$\tan\alpha_1 = 4\,a\frac{y_1 - y}{y_1^2 - y^2} = \frac{4\,a}{y_1 + y}.$$

If we now let $P_1$ come to coincidence with $P$ so that $y_1$ becomes $= y$, we find for the ***slope of the tangent*** at $P(x, y)$:

$$\tan \alpha = \frac{2a}{y}. \tag{7}$$

This slope of the tangent at $P$ is also called the ***slope of the parabola*** at $P$. The ordinate $y$ of the parabola is a function of the abscissa $x$; and the slope of the parabola at $P(x, y)$ is the rate at which $y$ increases with increasing $x$ at $P$; in other words, it is the ***derivative*** $y'$ of $y$ with respect to $x$ (compare § 73).

As by the equation of the parabola we have $y = \pm 2\sqrt{ax}$, we find:

$$y' = \tan \alpha = \frac{2a}{y} = \pm \sqrt{\frac{a}{x}}. \tag{8}$$

The double sign in the last expression corresponds to the fact that to a given value of $x$ belong two points of the curve with equal and opposite slopes.

**98. Equation of the Tangent.** As the slope of the parabola

$$y^2 = 4ax$$

at the point $P(x, y)$ is $2a/y$ (§ 97), ***the equation of the tangent*** at this point is

$$Y - y = \frac{2a}{y}(X - x),$$

where $X$, $Y$ are the coordinates of any point of the tangent, while $x$, $y$ are the coordinates of the point of contact. This equation can be simplified by multiplying both sides by $y$ and observing that $y^2 = 4ax$; we thus find

$$yY = 2a(x + X). \tag{9}$$

Notice that (as in the case of the circle, § 54) the equation of the tangent is obtained from the equation of the curve, $y^2 = 4ax$, by replacing $y^2$ by $yY$, $2x$ by $x + X$.

The segment $TP$ (Fig. 58) of the tangent from its intersection $T$ with the axis of the parabola to the point of contact $P$ is called the *length of the tangent at P*; the projection $TQ$ of this segment $TP$ on the axis of the parabola is called the ***subtangent*** at $P$. Now, with $Y = 0$, equation (9) gives $X = -x$, *i.e.* $TO = OQ$; hence *the subtangent is bisected by the vertex.* This furnishes a simple construction for the tangent at any *point* $P$ of the parabola if the axis and vertex of the parabola are known.

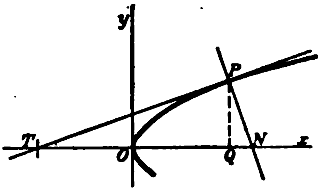

FIG. 58

**99. Equation of the Normal.** The ***normal*** at a point $P$ of any plane curve is defined as the perpendicular to the tangent through the point of contact.

The slope of the normal is therefore (§ 27) minus the reciprocal of that of the tangent. Hence the ***equation of the normal*** to the parabola is:

$$Y - y = -\frac{y}{2a}(X - x),$$

that is:

$$(10) \qquad yX + 2aY = (2a + x)y.$$

The segment $PN$ of the normal from the point $P(x, y)$ on the curve to the intersection $N$ of the normal with the axis of the parabola is called the *length of the normal* at $P$; the projection $QN$ of this segment $PN$ on the axis of the parabola is called the ***subnormal*** at $P$.

Now, with $Y = 0$, equation (10) gives $X = 2a + x$, and as $x = OQ$, it follows that $QN = 2a$; *i.e. the subnormal of the parabola is constant,* viz. equal to half the latus rectum.

**100. Intersections of a Line and a Parabola.** The intersections of the parabola

$$y^2 = 4\,ax$$

with the straight line

$$y = mx + b$$

are found by substituting the value of $y$ from the latter in the former equation:

$$(mx + b)^2 = 4\,ax,$$

or, reducing:

$$m^2x^2 + 2\,(mb - 2\,a)\,x + b^2 = 0.$$

The roots of this quadratic in $x$ are the abscissas of the points of intersection; the ordinates are then found from

$$y = mx + b.$$

It thus appears that *a straight line cannot intersect a parabola in more than two points.* If the roots are imaginary, the line does not meet the parabola; if they are real and equal, the line has but one point in common with the parabola and is *a tangent to the parabola* (provided $m \neq 0$).

**101. Slope Equation of the Tangent.** The condition for equal roots is

$$(bm - 2\,a)^2 = b^2m^2,$$

which reduces to

$$m = \frac{a}{b}.$$

The point that the line of this slope has in common with the parabola is then found to have the coordinates

$$x = \frac{2\,a - bm}{m^2} = \frac{b^2}{a},\ y = mx + b = 2\,b.$$

As the slope of the parabola at any point $(x, y)$ is (§ 97) $y' = 2\,a/y$, the slope at the point just found is $y' = a/b = m$; *i.e.* the slope of the parabola is the same as that of the line $y = mx + b$; this line is therefore a tangent. Thus, *the line*

$$(11) \qquad y = mx + \frac{a}{m}$$

*is tangent to the parabola* $y^2 = 4ax$ whatever the value of $m$.

This may be called the *slope-form of the equation of the tangent.* Equation (11) can also be deduced from the equation (9), by putting $2a/y = m$ and observing that $y^2 = 4ax$.

**102. Slope Equation of the Normal.** The equation (10) of the normal can be written in the form

$$Y = -\frac{y}{2a}X + y + \frac{xy}{2a},$$

or since by the equation (3) of the parabola $x = y^2/4a$:

$$Y = -\frac{y}{2a}X + y + \frac{y^3}{8a^2}.$$

If we denote by $n$ the slope of this normal, we have:

$$n = -\frac{y}{2a}, \quad y = -2an, \quad \frac{y^3}{8a^2} = -an^3,$$

so that the equation of the normal assumes the form

$$(12) \qquad Y = nX - 2an - an^3.$$

This may be called the *slope-form of the equation of the normal.*

**103. Tangents from an Exterior Point.** The slope-form (11) of the tangent shows that *from any point* $(x, y)$ *of the plane not more than two tangents can be drawn to the parabola* $y^2 = 4ax$. For, the slopes of these tangents are found by substituting in (11) for $x$, $y$ the coordinates of the given point and solving the resulting quadratic in $m$. This quadratic may have real and different, real and equal, or complex roots.

Those points of the plane for which the roots are real and different are said to lie *outside* the parabola; those points for which the roots are imaginary are said to lie *within* the parabola; those points for which the roots are equal lie *on* the parabola. The quadratic in $m$ can be written

$$xm^2 - ym + a = 0,$$

so that the discriminant is $y^2 - 4ax$. Therefore *a point* $(x, y)$ *of the plane lies within, on, or outside the parabola according as* $y^2 - 4ax$ *is less than, equal to, or greater than zero.*

Similarly, the slope-form (12) of the normal shows that *not more than three normals can be drawn from any point of the plane to the parabola,* since the equation (12) is a cubic for $n$ when the coordinates of any point of the plane are substituted for $X$, $Y$. As a cubic has always at least one real root there always exists one normal through a given point; but there may be two or three.

**104. Geometric Properties.** Let the tangent and normal at $P$ (Fig. 59) meet the axis at $T$, $N$; let $Q$ be the foot of the perpendicular from $P$ to the axis, $D$ that of the perpendicular to the directrix $d$; and let $O$ be the vertex, $F$ the focus.

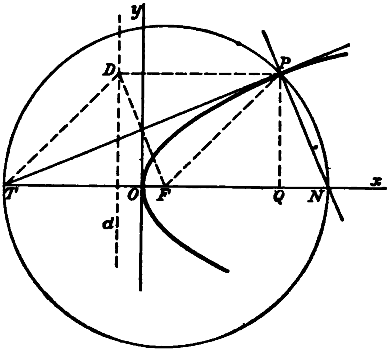

FIG. 59

As the subtangent $TQ$ is bisected by $O$ (§ 98) and the subnormal $QN$ is equal to $2a$ (§ 99), while $OF = a$, it follows that $F$ lies midway between $T$ and $N$.

The triangle $TPN$ being right-angled at $P$, and $F$ being the midpoint of its hypotenuse, it follows that

$$FP = FT = FN.$$

Hence, if axis and focus are given, the tangent and the normal at any point $P$ of the parabola are found by describing about $F$ a circle through $P$ which will meet the axis at $T$ and $N$.

As $FP = DP$, it follows that $FPDT$ is a rhombus; the diagonals $PT$ and $FD$ bisect therefore the angles of the rhombus and intersect at right angles. As $TP$ (like $TQ$) is bisected by the tangent at the vertex, the intersection of these diagonals lies on this tangent at the vertex. The properties just proved that *the tangent at P bisects the angle between the focal radius PF and the parallel PD to the axis* and that *the perpendicular from the focus to the tangent meets the tangent on the tangent at the vertex* are of particular importance.

**105. Diameters.** It is known from elementary geometry that in a circle all chords parallel to any given direction have their midpoints on a straight line which is a diameter of the circle.

Similarly, in a parabola, *the locus of the midpoints of all chords parallel to any given direction is a straight line*, and this line which is parallel to the axis is called a ***diameter*** of the parabola. To prove this, take the vertex as origin and the axis of the parabola as axis $Ox$ (Fig. 60) so that the equation is $y^2 = 4ax$. Any line of given slope $m$ has the equation

$$y = mx + b,$$

and with variable $b$ this represents a pencil of parallel lines. Eliminating $x$ we find for $y$ the quadratic

$$y^2 - \frac{4a}{m} y + \frac{4ab}{m} = 0.$$

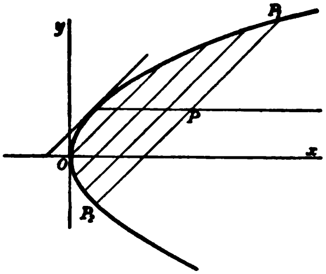

FIG. 60

The roots $y_1$, $y_2$ are the ordinates of the points $P_1$, $P_2$ at which the line intersects the parabola. The sum of the roots is

$$y_1 + y_2 = \frac{4a}{m};$$

hence the ordinate $\frac{1}{2}(y_1 + y_2)$ of the midpoint $P$ between $P_1$, $P_2$ is constant (*i.e.* independent of $x$), viz. $= 2a/m$, and independent of $b$. The midpoints of all chords of the same slope $m$ lie, therefore, on a parallel to the axis, at the distance $2a/m$ from it. The condition for equal roots (§ 101) gives $b = a/m$. That one of the parallels which passes through the point where the diameter meets the parabola is, therefore,

$$y = mx + \frac{a}{m};$$

by § 101 this is a tangent. Thus, ***the tangent at the end of a diameter is parallel to the chords bisected by the diameter.***

### EXERCISES

**1.** Find and sketch the tangent and normal of the following parabolas at the given points:

(*a*) $2y^2 = 25x$, $(2, 5)$. (*b*) $3y^2 = 4x$, $(3, -2)$. (*c*) $y^2 = 2x$, $(\frac{1}{2}, 1)$.
(*d*) $5y^2 = 12x$, $(\frac{5}{3}, -2)$. (*e*) $y^2 = x$, $(1, 1)$. (*f*) $45y^2 = x$, $(5, \frac{1}{3})$.

**2.** Show that the secant through the points $P(x, y)$ and $P_1(x_1, y_1)$ of the parabola $y^2 = 4ax$ has the equation $4aX - (y + y_1)Y + yy_1 = 0$, and that this reduces to the tangent at $P$ when $P_1$ and $P$ coincide.

**3.** Find the angle between the tangents to a parabola at the vertex and at the end of the latus rectum. Show that the tangents at the ends of the latus rectum are at right angles.

**4.** Find the length of the tangent, subtangent, normal, and subnormal of the parabola $y^2 = 4x$ at the point $(1, 2)$.

**5.** Find and sketch the tangents to the parabola $y^2 = 8x$ from each of the following points:

(*a*) $(-2, 3)$. (*b*) $(-2, 0)$. (*c*) $(-6, 0)$. (*d*) $(8, 8)$.

6. Draw the tangents to the parabola $y^2 = 3x$ that are inclined to the axis $Ox$ at the angles: ($a$) 30°, ($b$) 45°, ($c$) 135°, ($d$) 150°; and find their equations.

7. Find and sketch the tangents to the parabola $y^2 = 4x$ that pass through the point $(-2, 2)$.

8. Find and sketch the normals to the parabola $y^2 = 6x$ that pass through the points:

($a$) $(\frac{3}{2}, 0)$. ($b$) $(\frac{15}{2}, -3)$. ($c$) $(\frac{45}{8}, -\frac{9}{8})$. ($d$) $(\frac{9}{8}, -\frac{9}{8})$. ($e$) $(0, 0)$.

9. Are the following points inside, outside, or on the parabola $8y^2 = x$? ($a$) $(3, 1)$. ($b$) $(2, \frac{1}{2})$. ($c$) $(8, \frac{7}{8})$. ($d$) $(10, \frac{3}{2})$.

10. Show that any tangent to a parabola intersects the directrix and latus rectum (produced) in points equally distant from the focus.

11. Show that the tangents drawn to a parabola from any point of the directrix are perpendicular.

12. Show that the ordinate of the intersection of any two tangents to the parabola $y^2 = 4ax$ is the arithmetic mean of the ordinates of the points of contact, and the abscissa is the geometric mean of the abscissas of the points of contact.

13. Show that the sum of the slopes of any two tangents of the parabola $y^2 = 4ax$ is equal to the slope $Y/X$ of the radius vector of the point of intersection $(X, Y)$ of the tangents; find the product of the slopes.

14. Find the locus of the intersection of two tangents to the parabola $y^2 = 4ax$, if the sum of the slopes of the tangents is constant.

15. Find the locus of the intersection of two perpendicular tangents to a parabola; of two perpendicular normals to a parabola.

16. Show that the angle between any two tangents to a parabola is half the angle between the focal radii of the points of contact.

17. From the vertex of a parabola any two perpendicular lines are drawn; show that the line joining their other intersections with the parabola cuts the axis at a fixed point.

18. Find and sketch the diameter of the parabola $y^2 = 6x$ that bisects the chords parallel to $3x - 2y + 5 = 0$; give the equation of the focal chord of this system.

19. Find the system of parallel chords of the parabola $y^2 = 8x$ bisected by the line $y = 3$.

**20.** Show that the tangents at the extremities of any chord of a parabola intersect on the diameter bisecting this chord. Compare Ex. 12.

**21.** Find the length of the focal chord of a parabola of given slope $m$.

**22.** Find the angles at which the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$ intersect.

**23.** Two equal confocal parabolas have the same axis but open in opposite sense; show that they intersect at right angles.

**24.** If axis, vertex, and one other point of the parabola are given, additional points can be constructed as follows: Let $O$ be the vertex, $P$ the given point, and $Q$ the foot of the perpendicular from $P$ to the tangent at the vertex; divide $QP$ into equal parts by the points $A_1$, $A_2$, $\cdots$; and $OQ$ into the same number of equal parts by the points $B_1$, $B_2$, $\cdots$; the intersections of $OA_1$, $OA_2$, $\cdots$ with the parallels to the axis through $B_1$, $B_2$, $\cdots$ are points of the parabola.

**25.** If two tangents $AP_1$, $AP_2$ to a parabola with their points of contact $P_1$, $P_2$ are given and $AP_1$, $AP_2$ be divided into the same number of equal parts, the points of division being numbered from $P_1$ to $A$ and from $A$ to $P_2$, the lines joining the points bearing equal numbers are tangents to the parabola. To prove this show that the intersections of any tangent with the lines $AP_1$, $AP_2$ divide the segments $P_1A$, $AP_2$ in the same division ratio.

**26.** The shape assumed by a uniform chain or cable suspended between two fixed points $P_1$, $P_2$ is called a *catenary;* its equation is not algebraic and cannot be given here. But when the line $P_1P_2$ is nearly horizontal and the depth of the lowest point below $P_1P_2$ is small in comparison with $P_1P_2$, the catenary agrees very nearly with a parabola.

The distance between two telegraph poles is 120 ft.; $P_2$ lies 2 ft. above the level of $P_1$; and the lowest point of the wire is at 1/3 the distance between the poles. Find the equation of the parabola referred to $P_1$ as origin and the horizontal line through $P_1$ as axis $Ox$; determine the position of the lowest point and the ordinates at intervals of 20 ft.

**27.** The cable of a suspension bridge assumes the shape of a parabola if the weight of the suspended roadbed (together with that of the cables) is uniformly distributed horizontally. Suppose the towers of a bridge 240 ft. long are 60 ft. high and the lowest point of the cables is 20 ft. above the roadway; find the vertical distances from the roadway to the cables at intervals of 20 ft.

**28.** When a parabola revolves about its axis, it generates a surface called a paraboloid of revolution; all meridian sections (sections through the axis) are equal parabolas. If the mirror of a reflecting telescope is such a surface (the portion about the vertex), all rays of light falling in parallel to the axis are reflected to the same point; explain why.

**106. Parameter Equations.** Instead of using the cartesian or polar equation of a curve it is often more convenient to express $x$ and $y$ (or $r$ and $\phi$) each in terms of a third variable, which is then called the ***parameter***.

Thus the *parameter equations of a circle* of radius $a$ about the origin as center are:

$$x = a\cos\phi, \quad y = a\sin\phi,$$

$\phi$ being the parameter. To every value of $\phi$ corresponds a definite $x$ and a definite $y$, and hence a point of the curve. The elimination of $\phi$, by squaring and adding the equations, gives the cartesian equation $x^2 + y^2 = a^2$.

Again, to determine the *motion of a projectile* we may observe that, if gravity were not acting, the projectile, started with an initial velocity $v_0$ at an angle $\epsilon$ to the horizon, would have at the time $t$ the position

$$x = v_0\cos\epsilon\cdot t, \quad y = v_0\sin\epsilon\cdot t,$$

the horizontal as well as the vertical motion being uniform. But, owing to the constant acceleration $g$ of gravity (downward), the ordinate $y$ is diminished by $\frac{1}{2}gt^2$ in the time $t$, so that the coordinates of the projectile at the time $t$ are

$$x = v_0\cos\epsilon\cdot t, \quad y = v_0\sin\epsilon\cdot t - \tfrac{1}{2}gt^2.$$

These are the parameter equations of the path, the parameter here being the time $t$. The elimination of $t$ gives the cartesian equation of the parabola described by the projectile:

$$y = v_0\tan\epsilon\cdot x - \frac{g}{2\,v_0^2\cos^2\epsilon}x^2.$$

**107. Parameter Equations of a Parabola.** For any parabola $y^2 = 4\,ax$ we can also use as parameter the angle $\alpha$ made by the tangent with the axis $Ox$; we have for this angle (§ 97):

$$\tan\alpha = \frac{2\,a}{y};$$

it follows that $y = 2\,a\ \cot\alpha$ and hence $x = y^2/4\,a = a\cot^2\alpha$. The equations

$$x = a\cot^2\alpha, \quad y = 2\,a\cot\alpha$$

are *parameter equations of the parabola* $y^2 = 4\,ax$; the elimination of $\cot\alpha$ gives the cartesian equation.

**108. Parabola referred to Diameter and Tangent.** The equation of the parabola $y^2 = 4\,ax$ preserves this simple form if instead of axis and tangent at the vertex we take as axes any diameter and the tangent at its end. We shall show that the equation in these oblique coordinates is

$$y_1^2 = 4\,a_1x_1,$$

where $a_1$ is a new constant determined below.

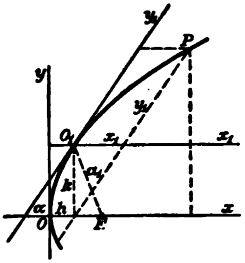

FIG. 61

To prove this observe that since the new origin $O_1$ $(h,\ k)$ is a point of the parabola $y^2 = 4\,ax$, we have by § 107

$$h = a\cot^2\alpha, \qquad k = 2\,a\cot\alpha,$$

where $\alpha$ is the angle at which the tangent at $O_1$ is inclined to the axis. Hence, transferring to parallel axes through $O_1$, we obtain the equation

$$(y + 2\,a\cot\alpha)^2 = 4\,a\,(x + a\cot^2\alpha),$$

which reduces to

$$y^2 + 4\,a\cot\alpha\cdot y = 4\,ax.$$

The relation between the rectangular coordinates $x$, $y$ and the oblique coordinates $x_1$, $y_1$, both with $O_1$ as origin, is readily seen from the figure to be $x = x_1 + y_1\cos\alpha$, $y = y_1\sin\alpha$. Substituting these values we find

$$y_1^2\sin^2\alpha + 4\,a\cos\alpha\cdot y_1 = 4\,ax_1 + 4\,ay_1\cos\alpha,$$

or, if we put $a/\sin^2\alpha = a_1$, $y_1^2 = 4\,\dfrac{a}{\sin^2\alpha}\,x_1 = 4\,a_1x_1$.

The meaning of the constant $a_1$ appears by observing that

$$a_1 = \frac{a}{\sin^2 \alpha} = \frac{1 + \tan^2 \alpha}{\tan^2 \alpha} a = a \cot^2 \alpha + a = h + a;$$

$a_1$ is therefore the distance of the new origin $O_1$ from the directrix, or, what amounts to the same, from the focus $F$.

### EXERCISES

1. Show that the parameter equations of a circle with center at $(h, k)$ and radius $a$ are

$$x = h + a \cos \phi, \quad y = k + a \sin \phi.$$

2. Sketch the curves whose equations are:

(*a*) $x = t, y = t^2$; (*b*) $x = t^2 - 1, y = 3 - 2t^2$;
(*c*) $x = 2t - 1, y = t^2 - 3t$; (*d*) $x = 3 + 2\cos \phi, y = 4 + 2\sin \phi$;
(*e*) $x = 4 + 5\cos \phi, y = 2 + 5\sin \phi$.

3. What must be the initial velocity $v_0$ of a projectile if, with an elevation of 30°, it is to strike an object 100 ft. above the horizontal plane of starting point at a horizontal distance from the latter of 1200 ft.?

4. What must be the elevation $e$ to strike an object 100 ft. above the horizontal plane of the starting point and 5000 ft. distant, if the initial velocity be 1200 ft. per second?

5. Prove that a projectile whose elevation is 60° rises three times as high as when its elevation is 30°, the magnitude of the initial velocity being the same in each case.

6. If a golf ball be driven from the tee horizontally with initial speed = 300 ft./sec., where and when would it land on ground 16 ft. below the tee if resistance of air and rotation of ball could be neglected?

7. A man standing 15 feet from a pole 150 ft. high aims at the top of the pole. If the bullet just misses the top, where will it strike the ground if $v_0 = 1000$ ft./sec.?

8. The ends $A$, $B$ of a straight rod of length $2a$ move along two perpendicular lines; find the locus of the midpoint of $AB$.

9. Four rods are jointed so as to form a parallelogram; if one side is fixed, find the path described by any point rigidly connected with the opposite side.

**109. Area of Parabolic Segment.** A parabola, together with any chord perpendicular to its axis, bounds an area $OPP'$ (shaded in Fig. 62). It was shown by Archimedes (about 250 B.C.) that this area is two thirds the area of the rectangle $PP'Q'Q$ that has the chord $P'P$ as one side and the tangent at the vertex as opposite side.

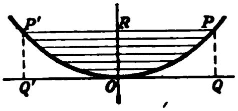

FIG. 62

This rectangle $PP'Q'Q$ is often called (somewhat improperly) the circumscribed rectangle so that the result can be expressed briefly by saying that *the area of the parabola is 2/3 of that of the circumscribed rectangle.*

This statement is of course equivalent to saying that *the* (non-shaded) *area* $OQP$ *is 1/3 of the area of the rectangle* $OQPR$. In this form the proposition is proved in the next article.

**110. Area by Approximation Process.** To obtain first an *approximate* value $(A)$ for the area $OQP$ (Fig. 63) we may subdivide the area into rectangular strips of equal width, by dividing $OQ$ into, say, $n$ equal parts and drawing the ordinates $y_1, y_2, \cdots y_n$. If the width of these strips is $\Delta x$ so that $OQ = n\Delta x$, we have as approximate value of the area:

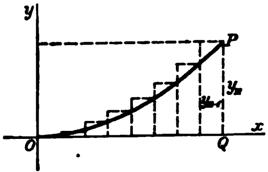

FIG. 63

$$(A) = \Delta x \cdot y_1 + \Delta x \cdot y_2 + \cdots + \Delta x \cdot y_n.$$

Now $y_1$ is the ordinate corresponding to the abscissa $\Delta x$; $y_2$ corresponds to the abscissa $2\,\Delta x$, etc.; $y_n$ corresponds to the abscissa $n\Delta x = OQ$. Hence, if the equation of the curve is $x^2 = 4\,ay$, we have:

$$y_1 = \frac{1}{4\,a}(\Delta x)^2, \quad y_2 = \frac{1}{4\,a}(2\,\Delta x)^2, \quad \cdots\; y_n = \frac{1}{4\,a}(n\Delta x)^2.$$

Substituting these values we find:

$$(A) = \frac{(\Delta x)^3}{4\,a}(1 + 2^2 + 3^2 + \cdots + n^2).$$

Now,

$$1 + 2^2 + \cdots + n^2 = \tfrac{1}{6}\,n(n+1)(2\,n+1) = \tfrac{1}{6}(2\,n^3 + 3\,n^2 + n);$$

hence
$$(A) = \frac{(\Delta x)^3}{24\,a}(2\,n^3 + 3\,n^2 + n) = \frac{(n\Delta x)^3}{24\,a}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right).$$

Now $n\Delta x = OQ = x_n$, the abscissa of the terminal point $P$, whatever the number $n$ and length $\Delta x$ of the subdivisions. Hence, if we let the number $n$ increase indefinitely, we find in the limit the *exact* expression $A$ for the area $OQP$:

$$A = \frac{x_n^3}{12\,a} = \frac{1}{3}x_n \cdot \frac{x_n^2}{4\,a} = \frac{1}{3}x_n y_n,$$

where $y_n = x_n^2/4\,a$ is the ordinate of the terminal point $P$. As $x_n y_n$ is the area of the rectangle $OQPR$, Archimedes' proposition (§ 109) is proved.

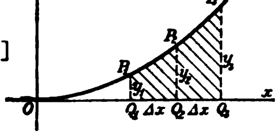

FIG. 64

**111. Area expressed in Terms of Ordinates.** The area (shaded in Fig. 64) between the parabola $x^2 = 4\,ay$, the axis $Ox$, and the two ordinates $y_1$, $y_3$, whose abscissas differ by $2\,\Delta x$ is evidently, by the formula of § 110,

$$A = \frac{1}{12\,a}(x_3^3 - x_1^3) = \frac{1}{12\,a}[(x_1 + 2\,\Delta x)^3 - x_1^3]$$

$$= \frac{\Delta x}{12\,a}(6\,x_1^2 + 12\,x_1\Delta x + 8\,(\Delta x)^2).$$

This expression can be given a remarkably simple form by introducing not only the ordinates $y_1 = x_1^2/4\,a$, $y_3 = (x_1 + 2\,\Delta x)^2/4\,a$, but also the ordinate $y_2$ midway between $y_1$ and $y_3$, whose abscissa is $x_1 + \Delta x$. For we have:

$$y_1 + 4\,y_2 + y_3 = \frac{1}{4\,a}[x_1^2 + 4(x_1 + \Delta x)^2 + (x_1 + 2\,\Delta x)^2]$$

$$= \frac{1}{4\,a}[6\,x_1^2 + 12\,x_1\Delta x + 8(\Delta x)^2].$$

We find therefore: $A = \frac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3)$.

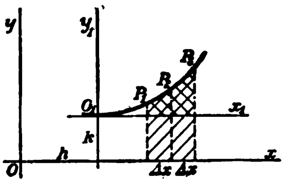

FIG. 65

This formula holds even if the vertex of the parabola is at any point $(h,\ k)$, provided the axis of the parabola is parallel to $Oy$. For (Fig. 65), to find the area under the arc $P_1P_2P_3$ we have only to add to the doubly shaded area the simply shaded rectangle whose area is $2\,k\Delta x$. We find therefore for the whole area:

$$\tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3) + 2\,k\Delta x = \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3 + 6\,k)$$

$$= \tfrac{1}{3}\,\Delta x\,[(y_1 + k) + 4\,(y_2 + k) + (y_3 + k)],$$

where $y_1$, $y_2$, $y_3$ are the ordinates of the parabola referred to its vertex, and hence $y_1 + k$, $y_2 + k$, $y_3 + k$ the ordinates for the origin $O$.

We have therefore for any parabola whose axis is parallel to $Oy$:

$$A = \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

**112. Approximation to any Area. Simpson's Rule.** The last formula is sometimes used to find an ***approximate value*** for the ***area under any curve*** (*i.e.* the area bounded by the axis $Ox$, an arc $AB$ of the curve, and the ordinates of $A$ and $B$, Fig. 66). This method is particularly convenient if a number of equidistant ordinates of the curve are known, or can be found graphically.

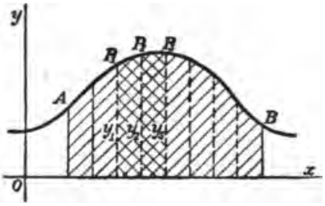

Fig. 66

Let $\Delta x$ be the distance of the ordinates, and let $y_1$, $y_2$, $y_3$ be any three consecutive ordinates. Then the doubly shaded portion of the required area, between $y_1$ and $y_3$, will be (if $\Delta x$ is sufficiently small) very nearly equal to the area under the parabola that passes through $P_1$, $P_2$, $P_3$ and has its axis parallel to $Oy$. This parabolic area is by § 111

$$= \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

The whole area under $AB$ is a sum of such expressions. This method for finding an approximate expression for the area under any curve is known as ***Simpson's rule*** (Thomas Simpson, 1743) although the fundamental idea of replacing an arc of the curve by a parabolic arc had been suggested previously by Newton.

**113. Area of any Parabolic Segment.** As the equation of a parabola referred to any diameter and the tangent at its end has exactly the same form as when the parabola is referred to its axis and the tangent at the vertex (§ 108) it can easily be shown that *the area of any parabolic segment is 2/3 of the area of the circumscribed parallelogram formed by the chord, the parallel tangent, and the two parallels to the axis through the extremities of the chord* (Fig. 67).

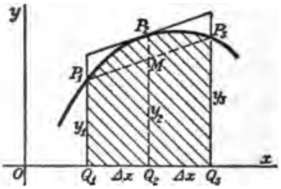

Fig. 67

With the aid of this proposition Simpson's rule can be proved very simply. For, the area of the parabolic segment $P_1P_3P_2$ (Fig. 67) is then equal to 2/3 of the parallelogram formed by the chord $P_1P_3$, the tangent at $P_2$, and the ordinates $y_1$, $y_3$ (produced if necessary). This parallelogram has a height $= 2\,\Delta x$ and a base $= MP_2 = y_2 - \frac{1}{2}(y_1 + y_3)$; hence the area of $P_1P_3P_2$ is $\frac{2}{3}\,\Delta x\,(2\,y_2 - y_1 - y_3) = \frac{1}{3}\,\Delta x\,[4\,y_2 - 2\,(y_1 + y_3)]$.

To find the whole shaded area we have only to add to this the area of the trapezoid $Q_1Q_3P_3P_1$, which is $\Delta x\,(y_1 + y_3)$.

$$\text{Hence } A = Q_1Q_3P_3P_2P_1 = \tfrac{1}{3}\,\Delta x[4\,y_2 - 2(y_1 + y_3) + 3(y_1 + y_3)]$$
$$= \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

## EXERCISES

**1.** Show that the area of any parabolic segment is 2/3 of the area of the circumscribed parallelogram.

**2.** In what ratio does the parabola $y^2 = 4\,ax$ divide the area of the circle $(x - a)^2 + y^2 = 4\,a^2$?

**3.** Find the area bounded by the parabola $y^2 = 4\,ax$ and a line of slope $m$ through the focus.

**4.** Find and sketch the curve whose ordinates represent the area bounded by: (*a*) the line $y = \frac{2}{3}\,x$, the axis $Ox$, and any ordinate, (*b*) the parabola $y = \frac{2}{3}\,x^2$, the axis $Ox$, and any ordinate.

**5.** Find an approximation to the areas bounded by the following curves and the axis $Ox$ (divide the interval in each case into eight or more equal parts):

(*a*) $4\,y = 16 - x^2$. (*b*) $y = (x + 3)(x - 2)^2$. (*c*) $y = x^2 - x^3$.

**6.** The cross-sections in square feet of a log at intervals of 6 ft. are 3.25, 4.27, 5.34, 6.02, 6.83; find the volume.

**7.** The cross-sections of a vessel in square feet measured at intervals of 3 ft. are 0, 2250, 5800, 8000, 10200; find the volume. Allowing one ton for each 35 cu. ft., what is the displacement of the vessel?

**8.** The half-widths in feet of a launch's deck at intervals of 5 ft. are 0, 1.8, 2.6, 3.2, 3.3, 3.3, 2.7, 2.1, 1; find the area.

# CHAPTER VII

## ELLIPSE AND HYPERBOLA

**114. Definition of the Ellipse.** The ***ellipse*** may be defined as *the locus of a point whose distances from two fixed points have a constant sum.*

If $F_1$, $F_2$ (Fig. 68) are the fixed points, which are called the ***foci***, and if $P$ is any point of the ellipse, the condition to be satisfied by $P$ is

$$F_1P + F_2P = 2a.$$

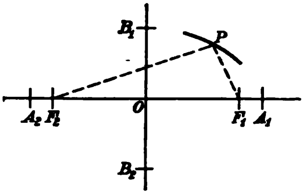

FIG. 68

The ellipse can be traced mechanically by attaching at $F_1$, $F_2$ the ends of a string of length $2a$ and keeping the string taut by means of a pencil. It is obvious that the curve will be symmetric with respect to the line $F_1F_2$, and also with respect to the perpendicular bisector of $F_1F_2$. These axes of symmetry are called the ***axes*** of the ellipse; their intersection $O$ is called the ***center*** of the ellipse.

**115. Axes.** The points $A_1$, $A_2$, $B_1$, $B_2$ (Figs. 68 and 69) where the ellipse intersects these axes are called ***vertices.*** The distance $A_2A_1$ of those vertices that lie on the axis containing the foci $F_1$, $F_2$ is $= 2a$, the length of the string. For when the point $P$ in describing the ellipse arrives at $A_1$, the string is doubled along $F_1A_1$ so that

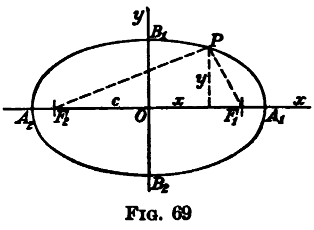

FIG. 69

$$F_2F_1 + 2\,F_1A_1 = 2\,a;$$

and since, by symmetry, $A_2F_2 = F_1A_1$, we have

$$A_2F_2 + F_2F_1 + F_1A_1 = A_2A_1 = 2\,a.$$

The distance $A_2A_1 = 2\,a$, which is called the ***major axis***, must evidently be not less than the distance $F_2F_1$ between the foci, which we shall denote by $2\,c$.

The distance $B_2B_1$ of the other two vertices is called the ***minor axis*** and will be denoted by $2\,b$. We then have

$$b^2 = a^2 - c^2;$$

for when $P$ arrives at $B_1$, we have $B_1F_2 = B_1F_1 = a$.

**116. Equation of the Ellipse.** If we take the center $O$ as origin and the axis containing the foci as axis $Ox$, the equation of the ellipse is readily found from the condition $F_1P + F_2P = 2\,a$, which gives, since the coordinates of the foci are $c$, 0 and $-c$, 0:

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2\,a.$$

Squaring both members we have

$$x^2 + y^2 + c^2 + \sqrt{(x^2 + y^2 + c^2 - 2\,cx)(x^2 + y^2 + c^2 + 2\,cx)} = 2\,a^2;$$

transferring $x^2 + y^2 + c^2$ to the right-hand member and squaring again, we find

$$(x^2 + y^2 + c^2)^2 - 4\,c^2x^2 = 4\,a^4 - 4\,a^2(x^2 + y^2 + c^2) + (x^2 + y^2 + c^2)^2,$$

*i.e.*

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Now for the ellipse (§ 115) $a^2 - c^2 = b^2$. Hence, dividing both members by $a^2b^2$, we find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

as the ***cartesian equation*** *of the ellipse referred to its axes.*

This equation shows at a glance: $(a)$ that the curve is symmetric to $Ox$ as well as to $Oy$; $(b)$ that the intercepts on the axes $Ox$, $Oy$ are $\pm\,a$, and $\pm\,b$. The lengths $a$, $b$ are called the *semi-axes.*

Solving the equation for $y$ we find

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}, \tag{2}$$

which shows that the curve does not extend beyond the vertex $A_1$ on the right, nor beyond $A_2$ on the left.

If $a$ and $b$ (or, what amounts to the same, $a$ and $c$) are given numerically, we can calculate from (2) the ordinates of as many points as we please. If, in particular, $a = b$ (and hence $c = 0$), the ellipse reduces to a *circle*.

### EXERCISES

**1.** Sketch the ellipse of semi-axes $a = 4$, $b = 3$, by marking the vertices, constructing the foci, and determining a few points of the curve from the property $F_1P + F_2P = 2a$. Write down the equation of this ellipse, referred to its axes.

**2.** Sketch the ellipse $x^2/16 + y^2/9 = 1$ by drawing the circumscribed rectangle and finding some points from the equation solved for $y$.

**3.** Sketch the ellipses: (*a*) $x^2 + 2y^2 = 1$. (*b*) $3x^2 + 12y^2 = 5$.
(*c*) $3x^2 + 3y^2 = 20$. (*d*) $x^2 + 20y^2 = 1$.

**4.** If in equation (1) $a < b$, the equation represents an ellipse whose foci lie on $Oy$. Sketch the ellipses:

(*a*) $\frac{x^2}{4} + \frac{y^2}{16} = 1$. (*b*) $20x^2 + y^2 = 1$. (*c*) $10x^2 + 9y^2 = 10$.

**5.** Find the equation of the ellipse referred to its axes when the foci are midpoints between the center and vertices.

**6.** Find the product of the slopes of chords joining any point of an ellipse to the ends of the major axis. What value does this product assume when the ellipse becomes a circle?

**7.** Derive the equation of the ellipse with foci at $(0, c)$, $(0, -c)$, and major axis $2a$.

**8.** Write the equations of the following ellipses: (*a*) with vertices at $(5, 0)$, $(-5, 0)$, $(0, 4)$, $(0, -4)$; (*b*) with foci at $(2, 0)$, $(-2, 0)$, and major axis 6.

**9.** Find the equation of the ellipse with foci at $(1, 1)$, $(-1, -1)$, and major axis 6, and sketch the curve.

**117. Definition of the Hyperbola.** The ***hyperbola*** can be defined as *the locus of a point whose distances from two fixed points have a constant difference.*

The fixed points $F_1$, $F_2$ are again called the ***foci;*** if $2a$ is the constant difference, every point $P$ of the hyperbola must satisfy the condition

$$F_2P - F_1P = \pm 2a.$$

Notice that the length $2a$ must here be not greater than the distance $F_2F_1 = 2c$ of the foci.

The curve is symmetric to the line $F_2F_1$ and to its perpendicular bisector.

A mechanism for tracing an arc of a hyperbola consists of a straightedge $F_2Q$ (Fig. 70) which turns about one of the foci, $F_2$; a string, of length $F_2Q - 2a$, is fastened to the straightedge at $Q$ and with its other end to the other focus, $F_1$. As the straightedge turns about $F_2$, the string is kept taut by means of a pencil at $P$ which describes the hyperbolic arc. Of course only a portion of the hyperbola can be traced in this manner.

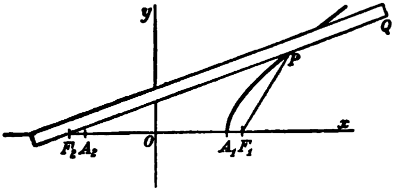

FIG. 70

**118. Equation of the Hyperbola.** If the line $F_2F_1$ be taken as the axis $Ox$, its perpendicular bisector as the axis $Oy$, and if $F_2F_1 = 2c$, the condition $F_2P - F_1P = \pm 2a$ becomes (Fig. 71):

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Squaring both members we find

$$x^2+y^2+c^2-\sqrt{(x^2+y^2+c^2-2\,cx)(x^2+y^2+c^2+2\,cx)}=2\,a^2;$$

squaring again and reducing as in § 116, we find exactly the same equation as in § 116:

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

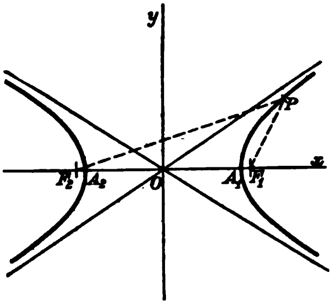

FIG. 71

But in the present case $c > a$, while for the ellipse we had $c < a$. We put, therefore, for the hyperbola

$$c^2-a^2=b^2;$$

the equation then reduces to the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{3}$$

which is the ***cartesian equation of the hyperbola referred to its axes.***

**119. Properties of the Hyperbola.** The equation (3) shows at once: (*a*) that the curve is symmetric to $Ox$ and to $Oy$; (*b*) that the intercepts on the axis $Ox$ are $\pm a$, and that the curve does not intersect the axis $Oy$.

The line $F_2F_1$ joining the foci and the perpendicular bisector of $F_2F_1$ are called the ***axes*** of the hyperbola; the intersection $O$ of these axes of symmetry is called the ***center.***

The hyperbola has only two ***vertices,*** viz. the intersections $A_1$, $A_2$ with the axis containing the foci.

The shape of the hyperbola is quite different from that of the ellipse. Solving the equation for $y$ we have

$$(4) \qquad y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that the curve extends to infinity from $A_1$ to the right and from $A_2$ to the left, but has no real points between the lines $x = a$, $x = -a$.

The line $F_2F_1$ containing the foci is called the ***transverse axis;*** the perpendicular bisector of $F_2F_1$ is called the ***conjugate axis.*** The lengths $a$, $b$ are called the *transverse and conjugate semi-axes.*

In the particular case when $a = b$, the equation (3) reduces to

$$x^2 - y^2 = a^2,$$

and such a hyperbola is called ***rectangular*** or ***equilateral.***

**120. Asymptotes.** In sketching the hyperbola (3) or (4) it is best to draw first of all the two straight lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

*i.e.*

$$(5) \qquad y = \pm \frac{b}{a} x,$$

which are called the ***asymptotes*** of the hyperbola.

Comparing with equation (4) it appears that, for any value of $x$, the ordinates of the hyperbola (4) are always (in absolute value) less than those of the lines (5); but the difference becomes less as $x$ increases, approaching zero as $x$ increases indefinitely.

Thus, the hyperbola approaches its asymptotes more and more closely, the farther we recede from the center on either side, without ever reaching these lines at any finite distance from the center.

## EXERCISES

**1.** Sketch the hyperbola $x^2/16 - y^2/4 = 1$, after drawing the asymptotes, by determining a few points from the equation solved for $y$; mark the foci.

**2.** Sketch the rectangular hyperbola $x^2 - y^2 = 9$. Why the name rectangular?

**3.** With respect to the same axes draw the hyperbolas:

(*a*) $20x^2 - y^2 = 12$. (*b*) $x^2 - 20y^2 = 12$. (*c*) $x^2 - y^2 = 12$.

**4.** The equation $-x^2/a^2 + y^2/b^2 = 1$ represents a hyperbola whose foci lie on the axis $Oy$. Sketch the curves:

(*a*) $-3x^2 + 4y^2 = 24$. (*b*) $x^2 - 3y^2 + 18 = 0$. (*c*) $x^2 - y^2 + 16 = 0$.

**5.** Sketch to the same axes the hyperbolas:

$$\frac{x^2}{9} - y^2 = 1, \quad \frac{x^2}{9} - y^2 = -1.$$

Two such hyperbolas having the same asymptotes, but with their axes interchanged, are called *conjugate.*

**6.** What happens to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as $a$ varies? as $b$ varies?

**7.** The equation $x^2/a^2 - y^2/b^2 = k$ represents a family of similar hyperbolas in which $k$ is the parameter. What happens as $k$ changes from 1 to $-1$? What members of this family are conjugate?

**8.** Find the foci of the hyperbolas:

(*a*) $9x^2 - 16y^2 = 144$. (*b*) $3x^2 - y^2 = 12$.

**9.** Find the hyperbola with foci $(0, 3)$, $(0, -3)$ and transverse axis 4.

**10.** Find the equation of the hyperbola referred to its axes when the distance between the vertices is one half the distance between the foci.

**11.** Find the distance from an asymptote to a focus of a hyperbola.

**12.** Show that the product of the distances from any point of a hyperbola to its asymptotes is constant.

**13.** Find the hyperbola through the point $(1, 1)$ with asymptotes

$$y = \pm 2x.$$

**14.** Find the equation of the hyperbola whose foci are $(1, 1)$, $(-1, -1)$, and transverse axis 2, and sketch the curve.

**121. Ellipse as Projection of Circle.** If a circle be turned about a diameter $A_2A_1 = 2a$ through an angle $\epsilon\,(< \frac{1}{2}\pi)$ and then projected on the original plane, the projection is an ellipse.

For, if in the original plane we take the center $O$ as origin and $OA_1$ as axis $Ox$ (Fig. 72), the ordinate $QP$ of every point $P$ of the projection is the projection of the corresponding ordinate $QP_1$ of the circle; *i.e.*

$$QP = QP_1 \cos \epsilon.$$

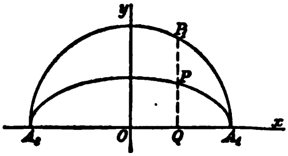

FIG. 72

The equation of the projection is therefore obtained from the equation

$$x^2 + y^2 = a^2$$

of the circle by replacing $y$ by $y/\cos \epsilon$. The resulting equation

$$x^2 + \frac{y^2}{\cos^2 \epsilon} = a^2$$

represents an ellipse whose semi-axes are $a$, the radius of the circle, and $b = a \cos \epsilon$, the projection of this radius.

**122. Construction of Ellipse from Circle.** We have just seen that, if $a > b$, the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be obtained from its *circumscribed circle* $x^2 + y^2 = a^2$ by reducing all the ordinates of this circle in the ratio $b/a$. This also appears by comparing the ordinates

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

of the ellipse with the ordinates $y = \pm \sqrt{a^2 - x^2}$ of the circle.

But the same ellipse can also be obtained from its *inscribed circle* $x^2 + y^2 = b^2$ by increasing each abscissa in the ratio $a/b$, as appears at once by solving for $x$.

It follows that when the semi-axes $a$, $b$ are given, points of the ellipse can be constructed by drawing concentric circles of radii $a$, $b$ and a pair of perpendicular diameters (Fig. 73); if any radius meets the circles at $P_1$, $P_2$, the intersection $P$ of the parallels through $P_1$, $P_2$ to the diameters is a point of the ellipse.

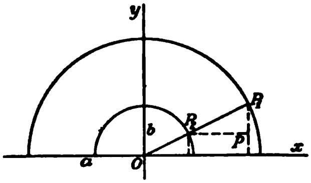

FIG. 73

**123. Tangent to Ellipse.** It follows from § 121 that if $P(x, y)$ is any point of the ellipse and $P_1$ that point of the circumscribed circle which has the same abscissa, *the tangents at $P$ to the ellipse and at $P_1$ to the circle must meet at a point $T$ on the major axis* (Fig. 74).

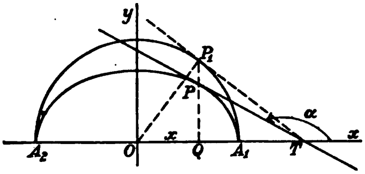

FIG. 74

For, as the circle is turned about $A_2A_1$ into the position in which $P$ is the projection of $P_1$, the tangent to the circle at $P_1$ is turned into the position whose projection is $PT$, the point $T$ on the axis remaining fixed.

The tangent $x_1X+y_1Y=a^2$ to the circle at $P_1(x_1, y_1)$ meets the axis $Ox$ at the point $T$ whose abscissa is

$$OT=a^2/x_1=a^2/x.$$

Hence the equation of the tangent at $P(x, y)$ to the ellipse is

$$\frac{X-\dfrac{a^2}{x}}{x-\dfrac{a^2}{x}}=\frac{Y}{y}, \text{ or } yX-\left(x-\frac{a^2}{x}\right)Y-a^2\frac{y}{x}=0;$$

dividing by $a^2y/x$ and observing that, by the equation of the ellipse, $x^2-a^2=-(a^2/b^2)y^2$ we find

$$\frac{xX}{a^2}+\frac{yY}{b^2}=1 \tag{6}$$

as *equation of the tangent to the ellipse* (1) *at the point* $P(x, y)$.

It follows from the equation of the tangent that the ***slope*** of the ellipse at any point $P(x, y)$ is

$$\tan\alpha=-\frac{b^2}{a^2}\frac{x}{y}.$$

**124. Eccentricity.** For the length of the *focal radius* $F_1P$ of any point $P(x, y)$ of the ellipse (1) we have (Fig. 75), since $a^2-b^2=c^2$:

$$F_1P^2=(x-c)^2+y^2=(x-c)^2+\frac{b^2}{a^2}(a^2-x^2)=\frac{1}{a^2}(a^4-2\,a^2cx+c^2x^2),$$

whence $\quad F_1P=\pm\left(a-\dfrac{c}{a}x\right).$

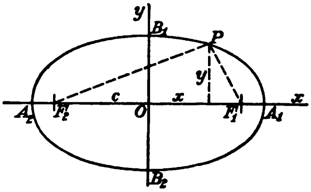

FIG. 75

The ratio $c/a$ of the distance $2\,c$ of the foci to the major axis $2\,a$ is called the (numerical) ***eccentricity*** of the ellipse. Denoting it by $e$ we have $F_1P=\pm(a-ex)$, and similarly we find $F_2P=\pm(a+ex)$.

For the hyperbola (3) we find in the same way, if we again put $e = c/a$, exactly the same expressions for the focal radii $F_1P$, $F_2P$ (in absolute value). But as for the ellipse $c^2 = a^2 - b^2$ while for the hyperbola $c^2 = a^2 + b^2$ it follows that *the eccentricity of the ellipse is always a proper fraction becoming zero only for a circle, while the eccentricity of the hyperbola is always greater than one.*

**125. Equation of Normal to Ellipse.** As the normal to a curve is the perpendicular to its tangent through the point of contact, the *equation of the normal* to the ellipse (1) at the point $P(x, y)$ is readily found from the equation (6) of the tangent as

$$\frac{y}{b^2}X - \frac{x}{a^2}Y = xy\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \frac{c^2}{a^2b^2}xy,$$

*i.e.*
$$\frac{a^2}{x}X - \frac{b^2}{y}Y = c^2.$$

The intercept made by this normal on the axis $Ox$ is therefore

$$ON = \frac{c^2}{a^2}x = e^2x.$$

From this result it appears by § 125 that (Fig. 76)

$$F_2N = c + e^2x = e(a + ex) = e \cdot F_2P,$$
$$F_1N = c - e^2x = e(a - ex) = e \cdot F_1P;$$

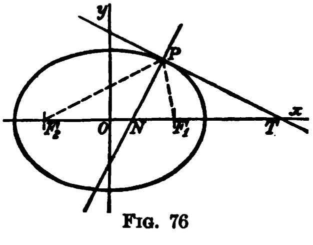

FIG. 76

hence the normal divides the distance $F_2F_1$ in the ratio of the adjacent sides $F_2P$, $F_1P$ of the triangle $F_1PF_2$. It follows that *the normal bisects the angle between the focal radii* $PF_1$, $PF_2$; in other words, the focal radii are equally inclined to the tangent.

**126. Construction of any Hyperbola from Rectangular Hyperbola.** The ordinates (4),

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

of the hyperbola (3) are $b/a$ times the corresponding ordinates

$$y = \pm \sqrt{x^2 - a^2}$$

of the equilateral hyperbola (end of § 119) having the same transverse axis. When $b < a$, we can put $b/a = \cos \epsilon$ and regard the general hyperbola as the projection of the equilateral hyperbola of equal transverse axis. When $b > a$, we can put $a/b = \cos \epsilon$ so that the equilateral hyperbola can be regarded as the projection of the general hyperbola.

In either case it is clear that the tangents to the general and equilateral hyperbolas at corresponding points (*i.e.* at points having the same abscissa) must intersect on the axis $Ox$.

**127. Slope of Equilateral Hyperbola.** To find the slope of the equilateral hyperbola

$$x^2 - y^2 = a^2,$$

observe that the slope of any secant joining the point $P(x, y)$ and $P_1(x_1, y_1)$ is $(y_1 - y)/(x_1 - x)$, and that the relations

$$y^2 = x^2 - a^2,\ y_1^2 = x_1^2 - a^2$$

give $\quad y^2 - y_1^2 = x^2 - x_1^2$, *i.e.* $(y - y_1)(y + y_1) = (x - x_1)(x + x_1)$,

whence
$$\frac{y - y_1}{x - x_1} = \frac{x + x_1}{y + y_1}.$$

Hence, in the limit when $P_1$ comes to coincidence with $P$, we find for the *slope of the tangent* at $P(x, y)$: $\tan \alpha = x/y$. Hence the equation of the tangent to the equilateral hyperbola is

$$Y - y = \frac{x}{y}(X - x), \text{ or } xX - yY = a^2,$$

since $x^2 - y^2 = a^2$.

**128. Tangent to the Hyperbola.** It follows as in § 123 that the *tangent to the general hyperbola* (3) has the equation

$$\frac{xX}{a^2}-\frac{yY}{b^2}=1. \tag{7}$$

The slope of the hyperbola (3) is therefore

$$\tan\alpha=\frac{b^2x}{a^2y}.$$

Notice that the equations (6), (7) of the tangents are obtained from the equations (1), (3) of the curves by replacing $x^2$, $y^2$ by $xX$, $yY$, respectively (compare §§ 54, 98).

It is readily shown (compare § 126) that for the hyperbola (3) the tangent meets the axis $Ox$ at the point $T$ that divides the distance of the foci $F_2F_1$ proportionally to the focal radii $F_2P$, $F_1P$, so that *the tangent to the hyperbola bisects the angle between the focal radii.*

### EXERCISES

1. Show that a right cylinder whose cross-section (*i.e.* section at right angles to the generators) is an ellipse of semi-axes $a$, $b$ has two (oblique) circular sections of radius $a$; find their inclinations to the cross-section.

2. Derive the equation of the normal to the hyperbola (3).

3. Find the polar equations of the ellipse and hyperbola, with the center as pole and the major (transverse) axis as polar axis.

4. Find the lengths of the tangent, subtangent, normal, and subnormal in terms of the coordinates at any point of the ellipse.

5. Show that an ellipse and hyperbola with common foci are orthogonal.

6. Show that the eccentricity of a hyperbola is equal to the secant of half the angle between the asymptotes.

7. Express the cosine of the angle between the asymptotes of a hyperbola in terms of its eccentricity.

8. Show that the tangents at the vertices of a hyperbola intersect the asymptotes at points on the circle about the center through the foci.

9. Show that the point of contact of a tangent to a hyperbola is the midpoint between its intersections with the asymptotes.

10. Show that the area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.

11. Show that the product of the distances from the center of a hyperbola to the intersections of any tangent with the asymptotes is constant.

12. Show that the tangent to a hyperbola at any point bisects the angle between the focal radii of the point.

13. As the sum of the focal radii of every point of an ellipse is constant (§ 116) and the normal bisects the angle between the focal radii (§ 125), a sound wave issuing from one focus is reflected by the ellipse to the other focus. This is the explanation of "whispering galleries." Find the semi-axes of an elliptic gallery in which sound is reflected from one focus to the other at a distance of 69 ft. in 1/10 sec. (the velocity of sound is 1090 ft./sec.).

14. Show that the distance from any point of an equilateral hyperbola to its center is a mean proportional to the focal radii of the point.

15. Show that the bisector of the angle formed by joining any point of an equilateral hyperbola to its vertices is parallel to an asymptote.

16. Show that the tangents at the extremities of any *diameter* (chord through the center) of an ellipse or hyperbola are parallel.

17. Let the normal at any point $P$ of an ellipse referred to its axes cut the coordinate axes at $Q$ and $R$; find the ratio $PQ/PR$.

18. Show that a tangent at any point of the circle circumscribed about an ellipse is also a tangent to the circle with center at a focus and radius equal to the focal radius of the corresponding point of the ellipse.

19. Show that the product of the $y$-intercept of the tangent at any point of an ellipse and the ordinate of the point of contact is constant.

20. Find the locus of the center of a circle which touches two fixed non-intersecting circles.

21. Find the locus of a point at which two sounds emitted at an interval of one second at two points 2000 ft. apart are heard simultaneously.

**129. Intersections of a Straight Line and an Ellipse.** The intersections of the ellipse (1) with any straight line are found by solving the simultaneous equations

$$b^2x^2 + a^2y^2 = a^2b^2,$$
$$y = mx + k.$$

Eliminating $y$, we find a quadratic equation in $x$:

$$(m^2a^2 + b^2)x^2 + 2\,mka^2x + (k^2 - b^2)a^2 = 0.$$

To each of the two roots the corresponding value of $y$ results from the equation $y = mx + k$.

Thus, *a straight line can intersect an ellipse in not more than two points.*

**130. Slope Form of Tangent Equations.** If the roots of the quadratic equation are equal, the line has but one point in common with the ellipse and is a tangent.

The condition for equal roots is

$$m^2k^2a^2 = (m^2a^2 + b^2)(k^2 - b^2),$$

whence $$k = \pm\sqrt{m^2a^2 + b^2}.$$

The two parallel lines

$$y = mx \pm \sqrt{m^2a^2 + b^2} \tag{8}$$

are therefore tangents to the ellipse (1), whatever the value of $m$. This equation is called the *slope form* of the equation of a tangent to the ellipse.

It can be shown in the same way that a straight line cannot intersect a hyperbola in more than two points, and that the two parallel lines

$$y = mx \pm \sqrt{m^2a^2 - b^2}$$

have each but one point in common with the hyperbola (3).

**131.** The condition that a line be a tangent to an ellipse or hyperbola assumes a simple form also when the line is given in the general form

$$Ax + By + C = 0.$$

Substituting the value of $y$ obtained from this equation in the equation (1) of the ellipse, we find for the abscissas of the points of intersection the quadratic equation:

$$(A^2a^2 + B^2b^2)x^2 + 2\,ACa^2x + (C^2 - B^2b^2)a^2 = 0;$$

the condition for equal roots is

$$A^2C^2a^2 = (A^2a^2 + B^2b^2)(C^2 - B^2b^2),$$

which reduces to

$$A^2a^2 + B^2b^2 = C^2.$$

The line is therefore a tangent whenever this condition is satisfied.

When the line is given in the normal form,

$$x \cos \beta + y \sin \beta = p,$$

the condition becomes

$$p^2 = a^2 \cos^2 \beta + b^2 \sin^2 \beta.$$

**132. Tangents from an Exterior Point.** By § 130 the line

$$y = mx + \sqrt{m^2a^2 + b^2}$$

is tangent to the ellipse (1) whatever the value of $m$. The condition that this line pass through any given point $(x_1, y_1)$ is

$$y_1 = mx_1 + \sqrt{m^2a^2 + b^2};$$

transposing the term $mx_1$, and squaring, we find the following quadratic equation for $m$:

$$m^2x_1^2 - 2\,mx_1y_1 + y_1^2 = m^2a^2 + b^2,$$

*i.e.*

$$(x_1^2 - a^2)m^2 - 2\,x_1y_1m + y_1^2 - b^2 = 0.$$

The roots of this equation are the slopes of those lines through $(x_1, y_1)$ that are tangent to the ellipse (1).

Thus, *not more than two tangents can be drawn to an ellipse from any point.* Moreover, these tangents are real and different, real and coincident, or imaginary, according as

$$x_1^2y_1^2 \overset{>}{\underset{<}{=}} (x_1^2 - a^2)(y_1^2 - b^2).$$

This condition can also be written in the form

$$b^2x_1^2 + a^2y_1^2 \gtreqless a^2b^2,$$

*i.e.*

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \gtreqless 0.$$

Hence, to see whether real tangents can be drawn from a point $(x_1, y_1)$ to the ellipse (1) we have only to substitute the coordinates of the point for $x$, $y$ in the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1;$$

if the expression is zero, the point $(x_1, y_1)$ lies on the ellipse, and only one tangent is possible; if the expression is positive, two real tangents can be drawn, and the point is said to lie *outside* the ellipse; if the expression is negative, no real tangents exist, and the point is said to lie *within* the ellipse.

These definitions of inside and outside agree with what we would naturally call the inside or outside of the ellipse. But the whole discussion applies equally to the hyperbola (3) where the distinction between inside and outside is not so obvious.

**133. Symmetry.** Since the ellipse, as well as the hyperbola, has two rectangular axes of symmetry, the *axes* of the curve, it has a ***center***, the intersection of these axes, *i.e.* a point of symmetry such that every chord through this point is bisected at this point (compare § 70). Analytically this means that since the equation (1), as well as (3), is not changed by replacing $x$ by $-x$, nor by replacing $y$ by $-y$, it is not changed by replacing both $x$ and $y$ by $-x$ and $-y$, respectively. In other words, if $(x, y)$ is a point of the curve, so is $(-x, -y)$. This fact is expressed by saying that the origin is a point of symmetry, or center.

**134. Conjugate Diameters.** Any chord through the center of an ellipse or hyperbola is called a ***diameter*** of the curve.

Just as in the case of the circle, so for the ellipse *the locus of the midpoints of any system of parallel chords is a diameter.* This follows from the corresponding property of the circle because the ellipse can be regarded as the projection of a circle (§ 121). But this diameter is in general not perpendicular to the parallel chords; it is said to be ***conjugate*** to the diameter that occurs among the parallel chords. Thus, in Fig. 77, $P'Q'$ is conjugate to $PQ$ (and *vice versa*).

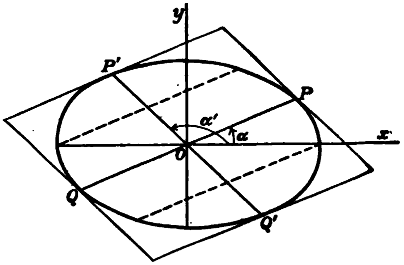

FIG. 77

To find the diameter conjugate to a given diameter $y = mx$ of the ellipse (1), let $y = mx + k$ be any parallel to the given diameter. If this parallel intersects the ellipse (1) at the real points $(x_1, y_1)$ and $(x_2, y_2)$, the midpoint has the coordinates $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$. The quadratic equation of § 129 gives

$$x = \frac{1}{2}(x_1 + x_2) = -\frac{ma^2k}{m^2a^2 + b^2}.$$

If instead of eliminating $y$ we eliminate $x$, we obtain the quadratic equation

$$(m^2a^2 + b^2)y^2 - 2kb^2y + (k^2 - m^2a^2)b^2 = 0,$$

whence
$$y = \frac{1}{2}(y_1 + y_2) = \frac{b^2k}{m^2a^2 + b^2}.$$

Eliminating $k$ between these results, we find the equation of the locus of the midpoints of the parallel chords of slope $m$:

$$(9) \qquad y = -\frac{b^2}{ma^2}x.$$

If $m = \tan \alpha$ is the slope of any diameter of the ellipse (1), *the slope of the conjugate diameter is*

$$m_1 = \tan \alpha_1 = -\frac{b^2}{ma^2}.$$

The diameter conjugate to this diameter of slope $m_1$ has therefore the slope

$$m_2 = -\frac{b^2}{m_1 a^2} = -\frac{b^2}{\left(-\frac{b^2}{ma^2}\right)a^2} = m;$$

*i.e.* it is the original diameter of slope $m$ (Fig. 77). In other words, either one of the diameters of slopes $m$ and $m_1$ is conjugate to the other; each bisects the chords parallel to the other.

**135. Tangents Parallel to Diameters.** Among the parallel lines of slope $m$, $y = mx + k$, there are two tangents to the ellipse, viz. (§ 130) those for which

$$k = \pm\sqrt{m^2a^2 + b^2},$$

their points of contact lie on (and hence determine) the conjugate diameter. This is obvious geometrically; it is readily verified analytically by showing that the coordinates of the intersections of the diameter of slope $-b^2/ma^2$ with the ellipse (1) satisfy the equations of the tangents of slope $m$, viz.

$$y = mx \pm \sqrt{m^2a^2 + b^2}.$$

The tangents at the ends of the diameter of slope $m$ must of course be parallel to the diameter of slope $m_1$. The four tangents at the extremities of any two conjugate diameters thus form a circumscribed parallelogram (Fig. 77).

The diameter conjugate to either axis of the ellipse is the other axis; the parallelogram in this case becomes a rectangle.

**136. Diameters of a Hyperbola.** For the hyperbola the same formulas can be derived except that $b^2$ is replaced throughout by $-b^2$. But the geometrical interpretation is somewhat different because a line $y = mx$ meets the hyperbola (3) in real points only when $m < b/a$.

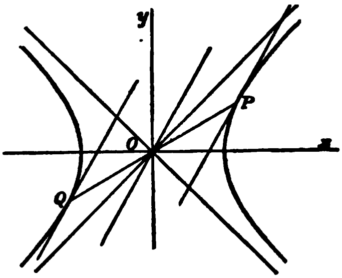

FIG. 78

The solution of the simultaneous equations

$$y = mx, \quad b^2x^2 - a^2y^2 = a^2b^2$$

gives:

$$x = \pm \frac{ab}{\sqrt{b^2 - m^2a^2}}, \quad y = \pm \frac{mab}{\sqrt{b^2 - m^2a^2}}.$$

These values are real if $m < b/a$ and imaginary if $m > b/a$ (Fig. 78). In the former case it is evidently proper to call the distance $PQ$ between the real points of intersection a *diameter* of the hyperbola; its length is

$$PQ = 2\sqrt{x^2 + y^2} = 2\,ab\sqrt{\frac{1 + m^2}{b^2 - m^2a^2}}.$$

If $m > b/a$, this quantity is imaginary; but it is customary to speak even in this case of a diameter, its length being defined as the real quantity

$$2\,ab\sqrt{\frac{1 + m^2}{m^2a^2 - b^2}}.$$

By this convention the analogy between the properties of the ellipse and hyperbola is preserved.

**137. Conjugate Diameters of a Hyperbola.** Two diameters of the hyperbola are called ***conjugate*** if their slopes $m$, $m_1$ are such that

$$mm_1 = \frac{b^2}{a^2}.$$

One of these lines evidently meets the curve in real points, the other does not.

If $m < b/a$, the line $y = mx$, as well as any parallel line, meets the hyperbola (3) in two real points, and the locus of the midpoints of the chords parallel to $y = mx$ is found to be the diameter conjugate to $y = mx$, viz.

$$y = m_1x = \frac{b^2}{ma^2}\,x.$$

If $m > b/a$, the coordinates $x_1$, $y_1$ and $x_2$, $y_2$ of the intersections of $y = mx$ with the hyperbola are imaginary; but the arithmetic means $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$ are real, and the locus of the points having these coordinates is the real line

$$y = m_1x = \frac{b^2}{ma^2}\,x.$$

It may finally be noted that what was in § 136 defined as the length of a diameter that does not meet the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in real points is the length of the real diameter of the hyperbola

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

two such hyperbolas are called *conjugate.*

**138. Parameter Equations. Eccentric Angle.** Just as the parameter equations of the circle $x^2 + y^2 = a^2$ are (§ 106):

$$x = a \cos \theta, \ y = a \sin \theta,$$

so those of the ellipse (1) are

$$x = a \cos \theta, \ y = b \sin \theta,$$

and those of the hyperbola (3) are

$$x = a \sec \theta, \ y = b \tan \theta.$$

In each case the elimination of the parameter $\theta$ (by squaring and then adding or subtracting) leads to the cartesian equation.

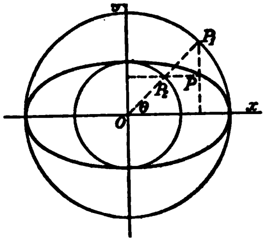

FIG. 79

The angle $\theta$, in the case of the circle, is simply the polar angle of the point $P(x, y)$. In the case of the ellipse, as appears from Fig. 79 (compare § 122), $\theta$ is the polar angle not of the point $P(x, y)$ of the ellipse, but of that point $P_1$ of the circumscribed circle which has the same abscissa as $P$, and also of that point $P_2$ of the inscribed circle which has the same ordinate as $P$. This angle $\theta = xOP_1$ is called the ***eccentric angle*** of the point $P(x, y)$ of the ellipse.

In the case of the hyperbola the eccentric angle $\theta$ determines the point $P(x, y)$ as follows (Fig. 80). Let a line through $O$ inclined at the angle $\theta$ to the transverse axis meet the circle of radius $a$ about the center at $A$, and let the transverse axis meet the circle of radius $b$ about the center at $B$. Let the tangent at $A$ meet the transverse axis at $A'$ and the tangent at $B$ meet the line $OA$ at $B'$. Then the parallels to the axes through $A'$ and $B'$ meet at $P$.

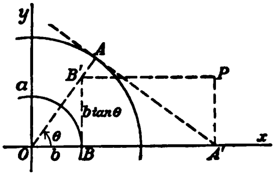

FIG. 80

**139. Area of Ellipse.** Since any ellipse of semi-axes $a$, $b$ can be regarded as the projection of a circle of radius $a$, inclined to the plane of the ellipse at an angle $\epsilon$ such that $\cos\epsilon = b/a$, the area $A$ of the ellipse is $A = \pi a^2 \cos\epsilon = \pi ab$.

### EXERCISES

1. Find the tangents to the ellipse $x^2 + 4y^2 = 16$, which pass through the following points:

$$(a)\ (2, \sqrt{3}),\quad (b)\ (-3, \tfrac{1}{2}\sqrt{7}),\quad (c)\ (4, 0),\quad (d)\ (-8, 0).$$

2. Find the tangents to the hyperbola $2x^2 - 3y^2 = 18$, which pass through the following points:

$$(a)\ (-6, 3\sqrt{2}),\quad (b)\ (-3, 0),\quad (c)\ (4, -\sqrt{5}),\quad (d)\ (0, 0).$$

3. Find the intersections of the line $x - 2y = 7$ and the hyperbola

$$x^2 - y^2 = 5.$$

4. Find the intersections of the line $3x + y - 1 = 0$ and the ellipse

$$x^2 + 4y^2 = 65.$$

5. For what value of $k$ will the line $y = 2x + k$ be a tangent to the hyperbola $x^2 - 4y^2 - 4 = 0$?

6. For what values of $m$ will the line $y = mx + 2$ be tangent to the ellipse $x^2 + 4y^2 - 1 = 0$?

7. Find the conditions that the following lines are tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$:

$$(a)\ Ax + By + C = 0,\quad (b)\ x\cos\beta + y\sin\beta = p.$$

8. Are the following points on, outside, or inside the ellipse $x^2 + 4y^2 = 4$?

$$(a)\ (\tfrac{3}{2}, \tfrac{3}{4}),\quad (b)\ (\tfrac{7}{4}, -\tfrac{1}{3}),\quad (c)\ (-\tfrac{1}{4}, -\tfrac{3}{4}).$$

9. Are the following points on, outside, or inside the hyperbola $9x^2 - y^2 = 9$? (a) $(\tfrac{5}{4}, -\tfrac{9}{4})$, (b) $(1.35, 2.15)$, (c) $(1.3, 2.6)$.

10. Find the difference of the eccentric angles of points at the extremities of conjugate diameters of an ellipse.

11. Show that conjugate diameters of an equilateral hyperbola are equal.

12. Show that an asymptote is its own conjugate diameter.

13. Show that the segments of any line between a hyperbola and its asymptotes are equal.

14. Find the tangents to an ellipse referred to its axes which have equal intercepts.

**15.** What is the greatest possible number of normals that can be drawn from a given point to an ellipse or hyperbola?

**16.** Show that tangents drawn at the extremities of any chord of an ellipse (or hyperbola) intersect on the diameter conjugate to the chord.

**17.** Show that lines joining the extremities of the axes of an ellipse are parallel to conjugate diameters.

**18.** Show that chords drawn from any point of an ellipse to the extremities of a diameter are parallel to conjugate diameters.

**19.** Find the product of the perpendiculars let fall to any tangent from the foci of an ellipse (or hyperbola).

**20.** The earth's orbit is an ellipse of eccentricity .01677 with the sun at a focus. The mean distance (major semi-axis) between the sun and earth is 93 million miles. Find the distance from the sun to the center of the orbit.

**21.** Find the sum of the squares of any two conjugate semi-diameters of an ellipse. Find the difference of the squares of conjugate semi-diameters of a hyperbola.

**22.** Find the area of the parallelogram circumscribed about an ellipse with sides parallel to any two conjugate diameters.

**23.** Find the angle between conjugate diameters of an ellipse in terms of the semi-diameters and semi-axes.

**24.** Express the area of a triangle inscribed in an ellipse referred to its axes in terms of the eccentric angles of the vertices.

**25.** The circle which is the locus of the intersection of two perpendicular tangents to an ellipse or hyperbola is called the *director-circle* of the conic. Find its equation: (*a*) For the ellipse. (*b*) For the hyperbola.

**26.** Find the locus of a point such that the product of its distances from the asymptotes of a hyperbola is constant. For what value of this constant is the locus the hyperbola itself?

**27.** Find the locus of the intersection of normals drawn at corresponding points of an ellipse and the circumscribed circle.

**28.** Two points $A$, $B$ of a line $l$ whose distance is $AB = a$ move along two fixed perpendicular lines; find the path of any point $P$ of $l$.

# CHAPTER VIII

## CONIC SECTIONS — EQUATION OF SECOND DEGREE

## PART I. DEFINITION AND CLASSIFICATION

**140. Conic Sections.** The ellipse, hyperbola, and parabola are together called ***conic sections***, or simply ***conics***, because the curve in which a right circular cone is intersected by any plane (not passing through the vertex) is an ellipse or hyperbola according as the plane cuts only one of the half-cones or both, and is a parabola when the plane is parallel to a generator of the cone. This will be proved and more fully discussed in §§ 148–152.

**141. General Definition.** The three conics can also be defined by a common property in the plane: *the locus of a point for which the ratio of its distances from a fixed point and from a fixed line is constant is a conic*, viz. an ***ellipse*** if the constant ratio is less than one, a ***hyperbola*** if the ratio is greater than one, and a ***parabola*** if the ratio is equal to one.

We shall find that this constant ratio is equal to the ***eccentricity*** $e = c/a$ as defined in § 124. Just as in the case of the parabola for which the above definition agrees with that of § 89, we shall call the fixed line $d_1$ ***directrix***, and the fixed point $F_1$ ***focus*** (Fig. 81).

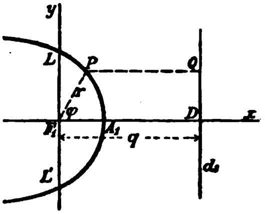

FIG. 81

**142. Polar Equation.** Taking the focus $F_1$ as pole, the perpendicular from $F_1$ toward the directrix $d_1$ as polar axis, and putting the given distance $F_1D = q$, we have $F_1P = r$, $PQ = q - r\cos\phi$, $r$ and $\phi$ being the polar coordinates of any point $P$ of the conic. The condition to be satisfied by the point $P$, viz. $F_1P/PQ = e$, *i.e.* $F_1P = e \cdot PQ$, becomes, therefore,

$$r = e(q - r\cos\phi), \text{ or } r = \frac{eq}{1 + e\cos\phi}.$$

This then is the ***polar equation of a conic if the focus is taken as pole and the perpendicular from the focus toward the directrix as polar axis.*** It is assumed that $q$ is not zero; the ratio $e$ may be any positive number.

**143. Plotting the Conic.** By means of this polar equation the conic can be plotted by points when $e$ and $q$ are given. Thus, for $\phi = 0$ and $\phi = \pi$, we find $eq/(1 + e)$ and $eq/(1 - e)$ as the intercepts $F_1A_1$ and $F_1A_2$ on the polar axis; $A_1$, $A_2$ are the vertices. For any negative value of $\phi$ (between 0 and $-\pi$) the radius vector has the same length as for the same positive value of $\phi$. The segment $LL'$ cut off by the conic on the perpendicular to the polar axis drawn through the pole is called the ***latus rectum;*** its length is $2\,eq$. Notice that in the ellipse and hyperbola, *i.e.* when $e \neq 1$, the vertex $A_1$ does not bisect the distance $F_1D$ (as it does in the parabola), but that

$$F_1A_1/A_1D = e.$$

If in Fig. 81, other things being equal, the sense of the polar axis be reversed, we obtain Fig. 82. We have again $F_1P = r$; but the distance of $P$ from the directrix $d_1$ is $QP = q + r\cos\phi$, so that the polar equation of the conic is now:

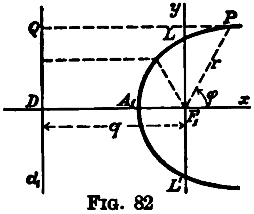

FIG. 82

$$r = \frac{eq}{1 - e\cos\phi}.$$

**144. Classification of Conics.** For $e = 1$, the equations of §§ 142–143 reduce to the equations of the parabola given in §§ 89, 90. It remains to show that for $e < 1$ and $e > 1$ these equations represent respectively an ellipse and a hyperbola as defined in §§ 114, 117.

To show this we need only introduce cartesian coordinates and then transform to the *center, i.e.* to the midpoint $O$ between the intersections $A_1$, $A_2$ of the curve with the polar axis.

**145. Transformation to Cartesian Coordinates.** The equation of § 142,

$$r = e(q - r \cos \phi)$$

becomes in cartesian coordinates, with the pole $F_1$ as origin and the polar axis as axis $Ox$ (Fig. 81):

$$\sqrt{x^2 + y^2} = e(q - x),$$

or, rationalized:

$$(1 - e^2)x^2 + 2\, e^2qx + y^2 = e^2q^2.$$

The midpoint $O$ between the vertices $A_1$, $A_2$ at which the curve meets the axis $Ox$ has, by § 143, the abscissa

$$\frac{1}{2}eq\left(\frac{1}{1+e} - \frac{1}{1-e}\right) = -\frac{e^2q}{1-e^2};$$

this also follows from the cartesian equation, with $y = 0$.

**146. Change of Origin to Center.** To transform to parallel axes through this point $O$ we have to replace $x$ by $x - e^2q/(1 - e^2)$; the equation in the new coordinates is therefore

$$(1 - e^2)\left(x - \frac{e^2q}{1-e^2}\right)^2 + 2\, e^2q\left(x - \frac{e^2q}{1-e^2}\right) + y^2 = e^2q^2,$$

and this reduces to

$$(1 - e^2)x^2 + y^2 = e^2q^2\left(1 + \frac{e^2}{1-e^2}\right) = \frac{e^2q^2}{1-e^2},$$

*i.e.*

$$\frac{x^2}{\dfrac{e^2q^2}{(1-e^2)^2}}+\frac{y^2}{\dfrac{e^2q^2}{1-e^2}}=1.$$

If $e<1$ this is an ellipse with semi-axes

$$a=\frac{eq}{1-e^2},\quad b=\frac{eq}{\sqrt{1-e^2}};$$

if $e>1$ it is a hyperbola with semi-axes

$$a=\frac{eq}{e^2-1},\quad b=\frac{eq}{\sqrt{e^2-1}}.$$

**147. Focus and Directrix.** The distance $c$ (in absolute value) from the center $O$ to the focus $F_1$ is, as shown above, for the ellipse

$$c=\frac{e^2q}{1-e^2}=ae,$$

for the hyperbola

$$c=\frac{e^2q}{e^2-1}=ae.$$

The distance (in absolute value) of the directrix from the center $O$ is for the ellipse, since $q=a(1-e^2)/e=a/e-ae$:

$$OD=c+q=ae+\frac{a}{e}-ae=\frac{a}{e},$$

and for the hyperbola, since $q=ae-a/e$:

$$OD=c-q=ae-ae+\frac{a}{e}=\frac{a}{e}.$$

It is clear from the symmetry of the ellipse and hyperbola that each of these curves has two foci, one on each side of the center at the distance $ae$ from the center, and two directrices whose equations are $x=\pm a/e$.

### EXERCISES

1. Sketch the following conics:

(*a*) $r=\dfrac{6}{2+3\cos\phi}$, (*b*) $r=\dfrac{2}{2+\cos\phi}$, (*c*) $r=\dfrac{1}{1-2\cos\phi}$.

**2.** Sketch the following conics and find their foci and directrices:

(a) $x^2 + 4y^2 = 4$, (b) $4x^2 + y^2 = 4$,
(c) $x^2 - 4y^2 = 4$, (d) $4x^2 - y^2 = 4$,
(e) $16x^2 + 25y^2 = 400$, (f) $9x^2 - 16y^2 = 144$,
(g) $9x^2 - 16y^2 + 144 = 0$, (h) $x^2 - y^2 = 2$.

**3.** Show that the following equations represent ellipses or hyperbolas and find their centers, foci, and directrices:

(a) $x^2 + 3y^2 - 2x + 6y + 1 = 0$, (b) $12x^2 - 4y^2 - 12x - 9 = 0$,
(c) $5x^2 + y^2 + 20x + 15 = 0$, (d) $5x^2 - 4y^2 + 8y + 16 = 0$.

**4.** Find the length of the latus rectum of an ellipse and a hyperbola in terms of the semi-axes.

**5.** Show that when tangents to an ellipse or hyperbola are drawn from any point of a directrix the line joining the points of contact passes through a focus.

**6.** From the definition (§ 141) of an ellipse and hyperbola, show that the sum and difference respectively of the focal radii of any point of the conic is constant.

**7.** Find the locus of the midpoints of chords drawn from one end of: (a) the major axis of an ellipse; (b) the minor axis.

**8.** Find the locus of § 141 when the fixed point lies on the fixed line.

**148. The Conics as Sections of a Cone.** As indicated by their name the conic sections, *i.e.* the parabola, ellipse, and hyperbola, can be defined as the curves in which a right circular cone is cut by a plane (§ 140).

In Figs. 83, 84, 85, $V$ is the vertex of the cone, $\measuredangle\, CVC' = 2\alpha$ the angle at its vertex; $OQ$ indicates the cutting plane, $CVC'$ that plane through the axis of the cone which is perpendicular to the cutting plane. The intersection $OQ$ of these two planes is evidently an axis of symmetry for the conic.

The conic is a parabola, ellipse, or hyperbola, according as $OQ$ is parallel to the generator $VC'$ of the cone (Fig. 83), meets $VC'$ at a point $O'$ belonging to the same half-cone as does $O$ (Fig. 84), or meets $VC'$ at a point $O'$ of the other half-cone (Fig. 85). If the angle $COQ$ be called $\beta$, the conic is

a parabola if $\beta = 2\,\alpha$ (Fig. 83),
an ellipse if $\beta > 2\,\alpha$ (Fig. 84),
a hyperbola if $\beta < 2\,\alpha$ (Fig. 85).

In each of the three figures $CC'$ represents the diameter $2\,r$ of any cross-section of the cone (*i.e.* of any section at right angles to its axis). We take $O$ as origin, $OQ$ as axis $Ox$, so that (Fig. 83) $OQ = x$, $QP = y$ are the coordinates of any point $P$ of the conic.

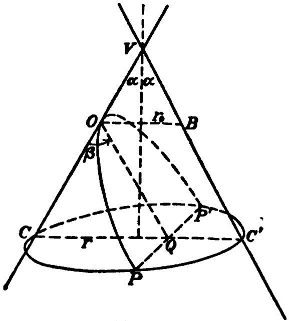

FIG. 83

As $QP$ is the ordinate of the circular cross-section $CPC'P'$ we have in each of the three cases $y^2 = QP^2 = CQ \cdot QC'$.

**149. Parabola.** In the first case (Fig. 83), when $\beta = 2\,\alpha$ so that $OQ$ is parallel to $VC'$, the expression

$$\frac{y^2}{x} = \frac{QP^2}{OQ} = \frac{CQ}{OQ} \cdot QC'$$

is constant, *i.e.* the same at whatever distance from the vertex we may take the cross-section $CPC'P'$. For, $QC'$ is equal to the diameter $OB = 2\,r_0$ of the cross-section through $O$, and $CQ/OQ = CC'/VC = 2\,r/r \csc \alpha = 2 \sin \alpha$. Hence, denoting the constant $r_0 \sin \alpha$ by $p$ we have

$$\frac{CQ}{OQ} \cdot QC' = 4\,r_0 \sin \alpha = 4\,p.$$

The equation of the conic in this case, referred to its axis $OQ$ and vertex $O$, is therefore $y^2 = 4\,px$. Notice that as $p = r_0 \sin \alpha$ the focus is found as the foot of the perpendicular from the midpoint of $OB$ on $OQ$.

**150. Ellipse.** In the second case (Fig. 84), *i.e.* when $\beta > 2\alpha$, if we put $OO' = 2a$, it can be shown that

$$\frac{y^2}{x(2a-x)} = \frac{QP^2}{OQ \cdot QO'}$$

is constant. For we have $QP^2 = CQ \cdot QC'$ and from the triangles $CQO$, $QC'O'$, observing that

$$\measuredangle\, QO'C' = \beta - 2\alpha:$$

$$\frac{CQ}{OQ} = \frac{\sin\beta}{\sin(\frac{1}{2}\pi - \alpha)},$$

$$\frac{QC'}{QO'} = \frac{\sin(\beta - 2\alpha)}{\sin(\frac{1}{2}\pi + \alpha)},$$

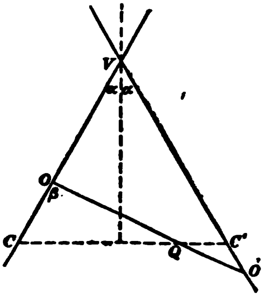

FIG. 84

whence

$$\frac{QP^2}{OQ \cdot QO'} = \frac{\sin\beta \sin(\beta - 2\alpha)}{\cos^2\alpha},$$

an expression independent of the position of the cross-section $CC'$. Denoting this positive constant by $k^2$, we find the equation

$$y^2 = k^2x(2a-x), \text{ or } \frac{(x-a)^2}{a^2} + \frac{y^2}{(ka)^2} = 1,$$

which is an ellipse, with semi-axes $a$, $ka$, and center $(a, 0)$.

**151. Hyperbola.** In the third case (Fig. 85), proceeding as in the second and merely observing that now $QO' = -(2a + x)$, we find the equation

$$y^2 = k^2x(2a + x),$$

*i.e.*
$$\frac{(x+a)^2}{a^2} - \frac{y^2}{(ka)^2} = 1,$$

which represents a hyperbola, with semi-axes $a$, $ka$ and center $(-a, 0)$.

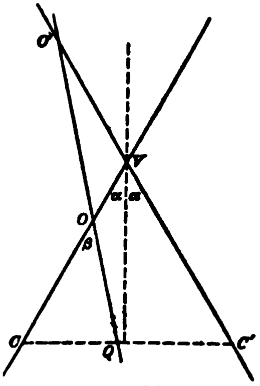

FIG. 85

**152. Limiting Cases.** The conic is an ellipse, hyperbola, or parabola according as $\beta > 2\alpha$, $< 2\alpha$, or $= 2\alpha$. Hence the *parabola* can be regarded as the limiting case of either an ellipse or a hyperbola whose center is removed to infinity.

If when $\beta > 2\alpha$ (Fig. 84), we let $\beta$ approach $\pi$, or if when $\beta < 2\alpha$ (Fig. 85), we let $\beta$ approach 0, the cutting plane becomes in the limit a tangent plane to the cone. It then has in common with the cone the points of the generator $VC$, and these only. A *single straight line* can thus appear as a limiting case of an ellipse or hyperbola.

Finally we obtain another class of limiting cases, or *cases of degeneration*, of the conics if, in any one of the three cases, we let the cutting plane pass through the vertex $V$ of the cone. In the first case, $\beta = 2\alpha$, the cutting plane is then tangent to the cone so that the parabola also may degenerate into a single straight line. In the second case, $\beta > 2\alpha$, if $\beta \neq \pi$, the ellipse degenerates into a single point, the vertex $V$ of the cone. In the third case, $\beta < 2\alpha$, if $\beta \neq 0$, the hyperbola degenerates into two intersecting lines. The term conic section, or *conic*, is often used as including these limiting cases.

## EXERCISES

1. For what value of $\beta$ in the preceding discussion does the conic become a circle?

2. Show that the spheres inscribed in a right circular cone so as to touch the cutting plane (Figs. 83, 84, 85) touch this plane at the foci of the conic.

3. The conic sections were originally defined (by the older Greek mathematicians, in the time of Plato, about 400 B.C.) as sections of a cone by a plane at right angles to a generator of the cone; show that the section is a parabola, ellipse, or hyperbola according as the angle $2\alpha$ at the vertex of the cone is $= \frac{1}{2}\pi$, $< \frac{1}{2}\pi$, $> \frac{1}{2}\pi$.

## PART II. REDUCTION OF GENERAL EQUATION

**153. Equations of Conics.** We have seen in the two preceding chapters that *by selecting the coordinate system in a convenient way* the equation of a *parabola* can be obtained in the simple form

$$y^2 = 4px,$$

that of an *ellipse* in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and that of a *hyperbola* in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

When the coordinate system is taken arbitrarily, the cartesian equations of these curves will in general not have this simple form; but they will always be of the second degree. To show this let us take the common definition of these curves (§ 141) as the locus of a point whose distances from a fixed point and a fixed line are in a constant ratio. With respect to any rectangular axes, let $x_1$, $y_1$ be the coordinates of the fixed point, $ax + by + c = 0$ the equation of the fixed line, and $e$ the given ratio. Then by §§ 9 and 42 the equation of the locus is

$$\sqrt{(x-x_1)^2 + (y-y_1)^2} = e \cdot \frac{ax + by + c}{\pm\sqrt{a^2 + b^2}},$$

or, rationalized:

$$(x - x_1)^2 + (y - y_1)^2 = \frac{e^2}{a^2 + b^2}(ax + by + c)^2.$$

It is readily seen that this equation is always of the second degree; *i.e.* that the coefficients of $x^2$, $y^2$, and $xy$ cannot all three vanish.

**154. Equation of Second Degree.** Conversely, *every equation of the second degree, i.e.* every equation of the form (§ 47)

(1) $$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

where $A$, $H$, $B$ are not all three zero, *in general represents a conic.* More precisely, the equation (1) may represent an ellipse, a hyperbola, or a parabola; it may represent two straight lines, different or coincident; it may be satisfied by the coordinates of only a single point; and it may not be satisfied by any real point.

Thus each of the equations

$$x^2 - 3\,y^2 = 0, \quad xy = 0$$

evidently represents two real different lines; the equation

$$x^2 - 2\,x + 1 = 0$$

represents a single line, or, as it is customary to say, two coincident lines; the equation

$$x^2 + y^2 = 0$$

represents a single point, while

$$x^2 + y^2 + 1 = 0$$

is satisfied by no real point and is sometimes said to represent an "imaginary ellipse."

The term *conic* is often used in a broader sense (compare § 152) so as to include all these cases; it is then equivalent to the expression "locus of an equation of the second degree."

It will be shown in the present chapter how to determine the locus of any equation of the form (1) with real coefficients. The method consists in selecting the axes of coordinates so as to reduce the given equation to its most simple form.

**155. Translation of Axes.** The transformation of the equation (1) to its most simple form is very easy in the particular case *when* (1) *contains no term in* $xy$, *i.e.* when $H = 0$. Indeed it suffices in this case *to complete the squares in* $x$ *and* $y$ *and transform to parallel axes.*

Two cases may be distinguished:

(*a*) $H = 0,\ A \neq 0,\ B \neq 0$, so that the equation has the form

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0. \tag{2}$$

Completing the squares in $x$ and $y$ (§ 48), we obtain an equation of the form

$$A\,(x - h)^2 + B\,(y - k)^2 = K,$$

where $K$ is a constant; upon taking parallel axes through the point $(h, k)$ it is seen that the locus is an ellipse, or a hyperbola, or two straight lines, or a point, or no real locus, according to the values of $A$, $B$, $K$.

(*b*) $H = 0$, and either $B = 0$ or $A = 0$, so that the equation is

$$Ax^2 + 2\,Gx + 2\,Fy + C = 0, \text{ or } By^2 + 2\,Gx + 2\,Fy + C = 0. \tag{3}$$

Completing the square in $x$ or $y$, we obtain

$$(x - h)^2 = p\,(y - k), \text{ or } (y - k)^2 = q\,(x - h);$$

with $(h, k)$ as new origin we have a parabola referred to vertex and axis, or two parallel lines, real and different, coincident, or imaginary.

It follows from this discussion that *the absence of the term in $xy$ indicates that, in the case of the ellipse or hyperbola, its axes, in the case of the parabola, its axis and tangent at the vertex, are parallel to the axes of coordinates.*

### EXERCISES

**1.** Reduce the following equations to standard forms and sketch the loci:

(*a*) $2\,y^2 - 3\,x + 8\,y + 11 = 0$,

(*b*) $x^2 + 4\,y^2 - 6\,x + 4\,y + 6 = 0$,

(*c*) $6\,x^2 + 3\,y^2 - 4\,x + 2\,y + 1 = 0$,

(*d*) $x^2 - 9\,y^2 - 6\,x + 18\,y = 0$,

(*e*) $9\,x^2 + 9\,y^2 - 36\,x + 6\,y + 10 = 0$,

(*f*) $2\,x^2 - 4\,y^2 + 4\,x + 4\,y - 1 = 0$,

(*g*) $x^2 + y^2 - 2\,x + 2\,y + 3 = 0$,

(*h*) $3\,x^2 - 6\,x + y + 6 = 0$,

(*i*) $x^2 - y^2 - 4\,x - 2\,y + 3 = 0$,

(*j*) $2\,x^2 - 5\,x + 12 = 0$,

(*k*) $2\,x^2 - 5\,x + 2 = 0$,

(*l*) $y^2 - 4\,y + 4 = 0$.

**2.** Find the equation of each of the following conics, determine the axis perpendicular to the given directrix, the vertices on this axis (by division-ratio), the lengths of the semi-axes, and make a rough sketch in each case:

(*a*) with $x - 2 = 0$ as directrix, focus at $(6, 3)$, eccentricity $\frac{1}{3}$;

(*b*) with $3x + 4y - 6 = 0$ as directrix, focus at $(5, 4)$, eccentricity $\frac{1}{2}$;

(*c*) with $x - y - 2 = 0$ as directrix, focus at $(4, 0)$, eccentricity $\frac{3}{2}$.

**3.** Find the axis, vertex, latus rectum, and sketch the parabola with focus at $(2, -2)$ and $2x - 3y - 5 = 0$ as directrix (see Ex. 2).

**4.** Prove the statement at the end of § 166.

**5.** Find the equation of the ellipse of major axis 5 with foci at $(0, 0)$ and $(3, 1)$.

**156. Rotation of Axes.** If the right angle $xOy$ formed by the axes $Ox$, $Oy$ be turned about the origin $O$ through an angle $\theta$ so as to take the new position $x_1Oy_1$ (Fig. 86), the relation between the old coordinates $OQ = x$, $QP = y$ of any point $P$ and the new coordinates $OQ_1 = x_1$, $Q_1P = y_1$ of the same point $P$ are seen from the figure to be

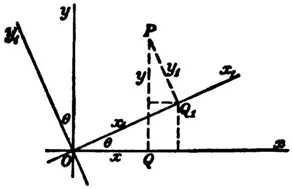

FIG. 86

$$(4) \qquad \begin{cases} x = x_1 \cos\theta - y_1 \sin\theta, \\ y = x_1 \sin\theta + y_1 \cos\theta. \end{cases}$$

By solving for $x_1$, $y_1$, or again from Fig. 86, we find

$$(4') \qquad \begin{cases} x_1 = x\cos\theta + y\sin\theta, \\ y_1 = -x\sin\theta + y\cos\theta. \end{cases}$$

If the cartesian equation of any curve referred to the axes

$Ox$, $Oy$ is given, the equation of the same curve referred to the new axes $Ox_1$, $Oy_1$ is found by substituting the values (4) for $x$, $y$ in the given equation.

**157. Translation and Rotation.** To transform from any rectangular axes $Ox$, $Oy$ (Fig. 87) to any other rectangular axes $O_1x_1$, $O_1y_1$, we have to combine the translation $OO_1$ (§ 13) with the rotation through an angle $\theta$ (§ 156).

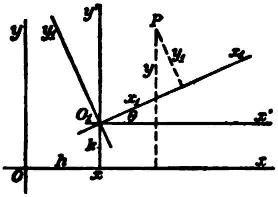

FIG. 87

This can be done by first transforming from $Ox$, $Oy$ to the parallel axes $O_1x'$, $O_1y'$ by means of the translation (§ 13)

$$x = x' + h,$$
$$y = y' + k,$$

and then turning the right angle $x'O_1y'$ through the angle $\theta = x'O_1x_1$, which is done by the transformation (§ 156)

$$x' = x_1 \cos\theta - y_1 \sin\theta,$$
$$y' = x_1 \sin\theta + y_1 \cos\theta.$$

Eliminating $x'$, $y'$, we find

$$(5) \qquad \begin{cases} x = x_1 \cos\theta - y_1 \sin\theta + h, \\ y = x_1 \sin\theta + y_1 \cos\theta + k. \end{cases}$$

The same result would have been obtained by performing first the rotation and then the translation.

It has been assumed that the right angles $xOy$ and $x_1Oy_1$ are *superposable;* if this were not the case, it would be necessary to invert ultimately one of the axes.

### EXERCISES

**1.** Find the coordinates of each of the following points after the axes have been rotated about the origin through the indicated angle:

(*a*) $(3, 4)$, $\frac{1}{4}\pi$. (*b*) $(0, 5)$, $\frac{1}{2}\pi$.
(*c*) $(-3, 2)$, $\theta = \tan^{-1}\frac{3}{2}$. (*d*) $(4, -3)$, $\frac{1}{3}\pi$.

**2.** If the origin is moved to the point $(2, -1)$ and the axes then rotated through 30°, what will be the new coordinates of the following points?

(*a*) $(0, 0)$. (*b*) $(2, 3)$. (*c*) $(6, -1)$.

**3.** Find the new equation of the parabola $y^2 = 4ax$ after the axes have been rotated through: (*a*) $\frac{1}{4}\pi$ , (*b*) $\frac{1}{2}\pi$ , (*c*) $\pi$ .

**4.** Show that the equation $x^2 + y^2 = a^2$ is not changed by any rotation of the axes about the origin. Why is this true?

**5.** Find the center of the circle $(x - a)^2 + y^2 = a^2$ after the axes have been turned about the origin through the angle $\theta$. What is the new equation?

**6.** For each of the following loci rotate the axes about the origin through the indicated angle and find the new equation:

(*a*) $x^2 - y^2 + 2 = 0$, $\frac{1}{4}\pi$. (*b*) $x^2 - y^2 = a^2$, $\frac{1}{2}\pi$.
(*c*) $y = mx + b$, $\theta = \tan^{-1} m$. (*d*) $12x^2 - 7xy - 12y^2 = 0$, $\theta = \tan^{-1}\frac{3}{4}$.
(*e*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{1}{2}\pi$. (*f*) $x^2 - y^2 = 0$, $\frac{1}{4}\pi$.

**7.** Through what angle must the axes be turned about the origin so that the circle $x^2 + y^2 - 3x + 4y - 5 = 0$ will not contain a linear term in $x$?

**8.** Suppose the right angle $x_1Oy_1$ (Fig. 89) turns about the origin at a uniform rate making one complete revolution in two seconds. The coordinates of a point with respect to the moving axes being $(2, 1)$, what are its coordinates with respect to the fixed axes $xOy$ at the end of: (*a*) $\frac{1}{2}$ sec.? (*b*) $\frac{2}{3}$ sec.? (*c*) 1 sec.? (*d*) $1\frac{1}{2}$ sec.?

**9.** In Fig. 89, draw the line $OP$, and denote $\angle QOP$ by $\phi$. Divide both sides of each of the equations (4) by $OP$ and show that they are then equivalent to the trigonometric formulas for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.

**158. Removal of the Term in $xy$.** The general equation of the second degree (1), § 154, when the axes are turned about the origin through an angle $\theta$ (§ 156), becomes:

$$\begin{aligned}&A(x_1\cos\theta - y_1\sin\theta)^2\\ &+2H(x_1\cos\theta - y_1\sin\theta)(x_1\sin\theta + y_1\cos\theta)\\ &+B(x_1\sin\theta + y_1\cos\theta)^2\\ &+2G(x_1\cos\theta - y_1\sin\theta)\\ &+2F(x_1\sin\theta + y_1\cos\theta) + C = 0.\end{aligned}$$

This is an equation of the second degree in $x_1$ and $y_1$ in which the coefficient of $x_1y_1$ is readily seen to be

$$\begin{aligned}-2A\cos\theta\sin\theta + 2B\sin\theta\cos\theta + 2H(\cos^2\theta - \sin^2\theta)&\\ = (B-A)\sin 2\theta + 2H\cos 2\theta.&\end{aligned}$$

It follows that if the axes be turned about the origin through an angle $\theta$ such that

$$(B-A)\sin 2\theta + 2H\cos 2\theta = 0,$$

*i.e.* such that

$$\tan 2\theta = \frac{2H}{A-B}, \qquad (6)$$

the equation referred to the new axes will contain no term in $x_1y_1$ and can therefore be treated by the method of § 155. According to the remark at the end of § 155 this means that the new axes $Ox_1$, $Oy_1$, obtained by turning the original axes $Ox$, $Oy$ through the angle $\theta$ found from (6), are parallel to the axes of the conic (or, in the case of the parabola, to the axis and the tangent at the vertex).

The equation (6) can therefore be used to determine *the directions of the axes of the conic;* but the process just indicated is generally inconvenient for reducing a numerical equation of the second degree to its most simple form since the values of $\cos\theta$ and $\sin\theta$ required by (4) to obtain the new equation are in general irrational.

### EXERCISES

1. Through what angle must the axes be turned about the origin to remove the term in $xy$ from each of the following equations?

(a) $3x^2+2\sqrt{3}xy+y^2-3x+4y-10=0$. (b) $x^2+2\sqrt{3}xy+7y^2-15=0$.

(c) $2x^2-3xy+2y^2+x-y+7=0$. (d) $xy=2a^2$.

2. Reduce each of the following equations to one of the forms in § 244.

(a) $xy=-2$. (b) $6x^2-5xy-6y^2=0$.

(c) $3x^2-10xy+3y^2+8=0$. (d) $13x^2-10xy+13y^2-72=0$.

**159. Transformation to Parallel Axes.** To transform the general equation of the second degree (1), § 154, to parallel axes through any point $(x_0, y_0)$, we have to substitute (§ 13)

$$x=x'+x_0, \quad y=y'+y_0,$$

the resulting equation is

$$Ax'^2+2Hx'y'+By'^2+2(Ax_0+Hy_0+G)x' \\ +2(Hx_0+By_0+F)y'+C'=0,$$

where the new constant term is

$$C'=Ax_0^2+2Hx_0y_0+By_0^2+2Gx_0+2Fy_0+C. \tag{7}$$

It thus appears that *after any translation of the coordinate system:*

(a) the coefficients of the terms of the second degree remain unchanged;

(b) the new coefficients of the terms of the first degree are linear functions of the coordinates of the new origin;

(c) the new constant term is the result of substituting the coordinates of the new origin in the left-hand member of the original equation.

**160. Transformation to the Center.** The transformed equation will contain no terms of the first degree, *i.e.* it will be of the form

$$(8)\qquad Ax'^2 + 2\,Hx'y' + By'^2 + C' = 0,$$

if we can select the new origin $(x_0, y_0)$ so that

$$(9)\qquad \begin{aligned} Ax_0 + Hy_0 + G &= 0,\\ Hx_0 + By_0 + F &= 0. \end{aligned}$$

This is certainly possible whenever

$$AB - H^2 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} \neq 0,$$

and we then find:

$$(10)\qquad x_0 = \frac{FH - GB}{AB - H^2},\quad y_0 = \frac{GH - FA}{AB - H^2}.$$

As the equation (8) remains unchanged when $x'$, $y'$ are replaced by $-x'$, $-y'$, respectively, the new origin so found is the *center* of the curve (§ 133). The locus is therefore in this case a *central conic, i.e.* an ellipse or a hyperbola; but it may reduce to two straight lines or to a point (see § 162). It might be entirely imaginary, viz. if $H = 0$; but the case when $H = 0$ has already been discussed in § 155.

We shall discuss in § 164 the case in which $AB - H^2 = 0$.

**161. The Constant Term and the Discriminant.** The calculation of the constant term $C'$ can be somewhat simplified by observing that its expression (7) can be written

$$C' = (Ax_0 + Hy_0 + G)x_0 + (Hx_0 + By_0 + F)y_0 + Gx_0 + Fy_0 + C,$$

*i.e.*, owing to (9),

$$(11)\qquad C' = Gx_0 + Fy_0 + C.$$

If we here substitute for $x_0$, $y_0$ their values (10) we find:

$$C' = \frac{GFH - G^2B + FGH - F^2A + ABC - H^2C}{AB - H^2}.$$

The numerator, which is called the ***discriminant*** of the equation of the second degree and is denoted by $D$, can be written in the form of a symmetric determinant, viz.

$$D=\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

If we denote the cofactors of this determinant by the corresponding small letters, we have $x_0=g/c$, $y_0=f/c$, $C'=D/c$. Notice that the coefficients of the equations (9), which determine the center, are given by the first two rows of $D$, while the third row gives the coefficients of $C'$ in (11).

**162. Straight Lines.** After transforming to the center, *i.e.* obtaining the equation (8), we must distinguish two cases according as $C'=0$ or $C'\neq 0$. The condition $C'=0$ means by (7) that the center lies on the locus; and indeed the homogeneous equation

$$Ax'^2+2Hx'y'+By'^2=0$$

represents two straight lines through the new origin $(x_0, y_0)$ (§ 45). The separate equations of these lines, referred to the new axes, are found by factoring the left-hand member. As we here assume (§ 160) that $AB-H^2\neq 0$, and $H\neq 0$, the lines can only be either real and different, or imaginary. In the latter case the point $(x_0, y_0)$ is the only real point whose coordinates satisfy the original equation.

**163. Ellipse and Hyperbola.** If $C'\neq 0$ we can divide (8) by $-C'$ so that the equation reduces to the form

$$(12) \qquad ax^2+2hxy+by^2=1.$$

This equation represents an ellipse or a hyperbola (since we assume $h\neq 0$). The axes of the ellipse or hyperbola can be found in magnitude and direction as follows.

If an ellipse or hyperbola, with its center, be given graphically, the axes can be constructed by intersecting the curve with a concentric circle and drawing the lines from the center to the intersections; the bisectors of the angles between these lines are evidently the axes of the curve (Fig. 88).

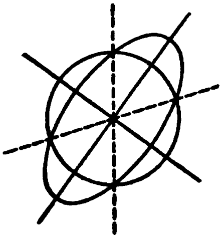

FIG. 88

The intersections of the curve (12) with a concentric circle of radius $r$ are given by the simultaneous equations

$$ax^2 + 2\,hxy + by^2 = 1, \qquad x^2 + y^2 = r^2;$$

dividing the second equation by $r^2$ and subtracting it from the first, we have

$$\left(a - \frac{1}{r^2}\right)x^2 + 2\,hxy + \left(b - \frac{1}{r^2}\right)y^2 = 0. \tag{13}$$

This *homogeneous* equation represents two straight lines through the origin, and as the equation is satisfied by the coordinates of the points that satisfy both the preceding equations, these lines must be the lines from the origin to the intersections of the circle with the curve (12). If we now select $r$ so as to make the two lines (13) coincide, they will evidently coincide with one or the other of the axes of the curve (12). The condition for equal roots of the quadratic (13) in $y/x$ is

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) - h^2 = 0. \tag{14}$$

This equation, which is quadratic in $1/r^2$ and can be written

$$\left(\frac{1}{r^2}\right)^2 - (a + b)\frac{1}{r^2} + ab - h^2 = 0, \tag{14'}$$

determines *the lengths of the axes.* If the two values found for $r^2$ are both positive, the curve is an ellipse; if one is positive and the other negative, it is a hyperbola; if both are negative, there is no real locus.

Each of the two values of $1/r^2$ found from (14′), if substituted in (13), makes the left-hand member, owing to (14), a complete square. *The equations of the axes* are therefore

$$\sqrt{a-\frac{1}{r^2}}\,x \pm \sqrt{b-\frac{1}{r^2}}\,y = 0,$$

or, multiplying by $\sqrt{a-1/r^2}$ and observing (14):

$$\left(a-\frac{1}{r^2}\right)x \pm hy = 0.$$

**164. Parabola.** It remains to discuss the case (§ 160) of the general equation of the second degree,

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

in which we have

$$AB - H^2 = 0.$$

This condition means that *the terms of the second degree form a perfect square:*

$$Ax^2 + 2\,Hxy + By^2 = (\sqrt{A}x + \sqrt{B}y)^2.$$

Putting $\sqrt{A} = a$ and $\sqrt{B} = b$ we can write the equation of the second degree in this case in the form

$$(15) \qquad (ax + by)^2 = -2\,Gx - 2\,Fy - C.$$

If $G$ and $F$ are both zero, this equation represents *two parallel straight lines,* real and different, real and coincident, or imaginary according as $C < 0$, $C = 0$, $C > 0$.

If $G$ and $F$ are not both zero, the equation (15) can be interpreted as meaning that the square of the distance of the point $(x, y)$ from the line

$$(16) \qquad ax + by = 0$$

is proportional to the distance of $(x, y)$ from the line

$$(17) \qquad 2\,Gx + 2\,Fy + C = 0.$$

Hence if these lines (16), (17) happen to be at right angles, the locus of (15) is a *parabola*, having the line (16) as axis and the line (17) as tangent at the vertex.

But even when the lines (16) and (17) are not at right angles the equation (15) can be shown to represent a parabola. For if we add a constant $k$ within the parenthesis and compensate the right-hand member by adding the terms $2\,akx + 2\,bky + k^2$, the locus of (15) is not changed; and in the resulting equation

(18) $(ax + by + k)^2 = 2(ak - G)x + 2(bk - F)y + k^2 - C$

we can determine $k$ so as to make the two lines

(19) $$ax + by + k = 0,$$

(20) $$2(ak - G)x + 2(bk - F)y + k^2 - C = 0$$

perpendicular. The condition for perpendicularity is

$$a(ak - G) + b(bk - F) = 0,$$

whence

(21) $$k = \frac{aG + bF}{a^2 + b^2}.$$

With this value of $k$, then, the lines (19), (20) are at right angles; and if (19) is taken as new axis $Ox$ and (20) as new axis $Oy$, the equation (18) reduces to the simple form

$$y^2 = px.$$

The constant $p$, *i.e.* the latus rectum of the parabola, is found by writing (18) in the form

$$\left(\frac{ax + by + k}{\sqrt{a^2 + b^2}}\right)^2 = \frac{2\sqrt{(ak - G)^2 + (bk - F)^2}}{a^2 + b^2} \cdot \frac{2(ak - G)x + 2(bk - F)y + k^2 - C}{2\sqrt{(ak - G)^2 + (bk - F)^2}};$$

hence

$$p = \frac{2}{a^2 + b^2}\sqrt{(ak - G)^2 + (bk - F)^2}.$$

Substituting for $k$ its value (21) we can reduce it to

$$p = \frac{2(aF - bG)}{(a^2 + b^2)^{\frac{3}{2}}}.$$

### EXERCISES

**1.** Find the equation of each of the following loci after transforming to parallel axes through the center:

(a) $3x^2 - 4xy - y^2 - 3x - 4y + 7 = 0.$
(b) $5x^2 + 6xy + y^2 + 6x - 4y - 5 = 0.$
(c) $2x^2 + xy - 6y^2 - 7x - 7y + 5 = 0.$
(d) $x^2 - 2xy - y^2 + 4x - 2y - 8 = 0.$

**2.** Find that diameter of the conic $3x^2 - 2xy - 4y^2 + 6x - 4y + 2 = 0$ (a) which passes through the origin, (b) which is parallel to each coordinate axis.

**3.** For what values of $k$ do the following equations represent straight lines? Find their intersections.

(a) $2x^2 - xy - 3y^2 - 6x + 19y + k = 0.$
(b) $kx^2 + 2xy + y^2 - x - y - 6 = 0.$
(c) $3x^2 - 4xy + ky^2 + 8y - 3 = 0.$
(d) $x^2 + 2y^2 + 6x - 4y + k = 0.$

**4.** Show that the equations of conjugate hyperbolas $x^2/a^2 - y^2/b^2 = \pm 1$ and their asymptotes $x^2/a^2 - y^2/b^2 = 0$, even after a translation and rotation of the axes, will differ only in the constant terms and that the constant term of the asymptotes is the arithmetic mean between the constant terms of the conjugate hyperbolas.

**5.** Find the asymptotes and the hyperbola conjugate to

$$2x^2 - xy - 15y^2 + x + 19y + 16 = 0.$$

**6.** Find the hyperbola through the point $(-2, 1)$ which has the lines $2x - y + 1 = 0$, $3x + 2y - 6 = 0$ as asymptotes. Find the conjugate hyperbola.

**7.** Show that the hyperbola $xy = a^2$ is referred to its asymptotes as coordinate axes. Find the semi-axes and sketch the curve. Find and sketch the conjugate hyperbola.

**8.** The volume of a gas under constant temperature varies inversely as the pressure (Boyle's law), *i.e.* $vp = c$. Sketch the curve whose ordinates represent the pressure as a function of the volume for different values of $c$; *e.g.* take $c = 1, 2, 3$.

**9.** Sketch the hyperbola $(x - a)(y - b) = c^2$ and its asymptotes. Interpret the constants $a$, $b$, $c$ geometrically.

**10.** Sketch the hyperbola $xy + 3y - 6 = 0$ and its asymptotes.

**11.** Find the center and semi-axes of the following conics, write their equations in the most simple form, and sketch the curves:

(*a*) $5x^2 - 6xy + 5y^2 + 12\sqrt{2}x - 4\sqrt{2}y + 8 = 0.$

(*b*) $x^2 - 6\sqrt{3}xy - 5y^2 - 16 = 0.$ (*c*) $x^2 + xy + y^2 - 3y + 6 = 0.$

(*d*) $13x^2 - 6\sqrt{3}xy + 7y^2 - 64 = 0.$

(*e*) $2x^2 - 4xy + y^2 + 2x - 4y - \frac{5}{2} = 0.$

(*f*) $3x^2 + 2xy + y^2 + 6x + 4y + \frac{7}{2} = 0.$

**12.** Sketch the following parabolas:

(*a*) $x^2 - 2\sqrt{3}xy + 3y^2 - 6\sqrt{3}x - 6y = 0.$

(*b*) $x^2 - 6xy + 9y^2 - 3x + 4y - 1 = 0.$

**13.** Show that the following combinations of the coefficients of the general equation of the second degree are *invariants* (*i.e.* remain unchanged) under any transformation from rectangular to rectangular axes:

(*a*) $A + B.$ (*b*) $AB - H^2.$ (*c*) $(A - B)^2 + 4H^2.$

**14.** Show that $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ represents a parabola. Sketch the locus.

**15.** Find the parabola with $x + y = 0$ as directrix and $(\frac{1}{2}a, \frac{1}{2}a)$ as focus.

**16.** Let five points $A$, $B$, $C$, $D$, $E$ be taken at equal intervals on a line. Show that the locus of a point $P$ such that $AP \cdot EP = BP \cdot DP$ is an equilateral hyperbola. (Take $C$ as origin.)

**17.** The variable triangle $AQB$ is isosceles with a fixed base $AB$. Show that the locus of the intersection of the line $AQ$ with the perpendicular to $QB$ through $B$ is an equilateral hyperbola.

**18.** Let $A$ be a fixed point and let $Q$ describe a fixed line. Find the locus of the intersection of a line through $Q$ perpendicular to the fixed line and a line through $A$ perpendicular to $AQ$.

**19.** Find the locus of the intersection of lines drawn from the extremities of a fixed diameter of a circle to the ends of the perpendicular chords.

**20.** Show by (14′), § 163, that if the equation of the second degree represents an ellipse, parabola, hyperbola, we have, respectively,

$$AB - H^2 > 0, = 0, < 0.$$

# CHAPTER IX

## HIGHER PLANE CURVES

## PART I. ALGEBRAIC CURVES

**165. Cubics.** It has been shown (§ 30) that every equation of the first degree,

$$\begin{aligned} &a_0 \\ &+ a_1x + b_1y = 0, \end{aligned}$$

represents a *straight line;* and (§ 154) that every equation of the second degree,

$$\begin{aligned} &a_0 \\ &+ a_1x + b_1y \\ &+ a_2x^2 + b_2xy + c_2y^2 = 0, \end{aligned}$$

either represents a *conic* or is not satisfied by any real points.

The *locus* represented by an equation of the third degree,

$$\begin{aligned} &a_0 \\ &+ a_1x + b_1y \\ &+ a_2x^2 + b_2xy + c_2y^2 \\ &+ a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 = 0, \end{aligned}$$

*i.e.* the aggregate of all real points whose coordinates $x$, $y$ satisfy this equation, is called a ***cubic curve.***

Similarly, the locus of all points that satisfy any equation of the *fourth* degree is called a *quartic curve;* and the terms *quintic*, *sextic*, etc., are applied to curves whose equations are of the *fifth*, *sixth*, etc., degrees.

Even the cubics present a large variety of shapes; still more so is this true of higher curves. We shall not discuss such curves in detail, but we shall study some of their properties.

**166. Algebraic Curves.** The general form of an *algebraic equation of the nth degree in x and y* is

$$
\begin{array}{c}
a_0 \\
+ a_1x + b_1y \\
+ a_2x^2 + b_2xy + c_2y^2 \\
+ a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
+ a_nx^n + b_nx^{n-1}y + \cdots + k_nxy^{n-1} + l_ny^n = 0.
\end{array}
\tag{1}
$$

The coefficients are supposed to be any real numbers, those in the last line being not all zero. The number of terms is not more than $1 + 2 + 3 + \cdots + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$.

If the cartesian equation of a curve can be reduced to this form by rationalizing and clearing of fractions, the curve is called an *algebraic curve of degree n.*

An algebraic curve of degree $n$ can be intersected by a straight line,

$$Ax + By + C = 0,$$

in not more than $n$ points. For, the substitution in (1) of the value of $y$ (or of $x$) derived from the linear equation gives an equation in $x$ (or in $y$) of a degree not greater than $n$; this equation can therefore have not more than $n$ roots, and these roots are the abscissas (or ordinates) of the points of intersection.

We have already studied the curves that represent the polynomial function

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n;$$

such a curve is an algebraic curve, but it is readily seen by comparison with the preceding equation that this equation is of a very special type, since it contains no term of higher degree than one in $y$. Such a curve is often called a *parabolic curve of the nth degree.*

**167. Transformation to Polar Coordinates.** The cartesian equation (1) is readily transformed to polar coordinates by substituting

$$x = r\cos\phi, \quad y = r\sin\phi;$$

it then assumes the form:

$$
\begin{aligned}
(2) \qquad & a_0 \\
& + (a_1\cos\phi + b_1\sin\phi)r \\
& + (a_2\cos^2\phi + b_2\cos\phi\sin\phi + c_2\sin^2\phi)r^2 \\
& + (a_3\cos^3\phi + b_3\cos^2\phi\sin\phi + c_3\cos\phi\sin^2\phi + d_3\sin^3\phi)r^3 \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& + (a_n\cos^n\phi + b_n\cos^{n-1}\phi\sin\phi + \cdots + k_n\cos\phi\sin^{n-1}\phi + l_n\sin^n\phi)r^n \\
& = 0.
\end{aligned}
$$

If any particular value be assigned to the polar angle $\phi$, this becomes an equation in $r$ of a degree not greater than $n$. Its roots $r_1$, $r_2$, $\cdots$ represent the intercepts $OP_1$, $OP_2$, $\cdots$ (Fig. 89) made by the curve (2) on the line $y = \tan\phi \cdot x$. Some of these roots may of course be imaginary, and there may be equal roots.

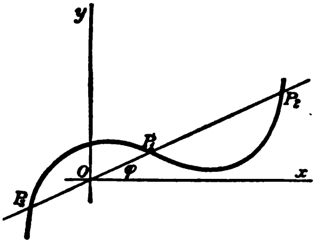

FIG. 89

**168. Curve through the Origin.** The equation in $r$ has at least one of its roots equal to zero if, and only if, the constant term $a_0$ is zero. Thus, *the necessary and sufficient condition that the origin O be a point of the curve is* $a_0 = 0$.

This is of course also apparent from the equation (1) which is satisfied by $x = 0$, $y = 0$ if, and only if, $a_0 = 0$.

**169. Tangent Line at Origin.** The equation (2) has at least two of its roots equal to zero if $a_0 = 0$ and $a_1\cos\phi + b_1\sin\phi = 0$. If $a_1$ and $b_1$ are not both zero, the latter condition can be satisfied by selecting the angle $\phi$ properly, viz. so that

$$\tan\phi = -\frac{a_1}{b_1}.$$

The line through the origin inclined at this angle $\phi$ to the polar axis is the *tangent to the curve at the origin* $O$ (Fig. 90). Its cartesian equation is $y = \tan\phi \cdot x = -(a_1/b_1)x$, *i.e.*

$$(3) \qquad a_1x + b_1y = 0.$$

Thus, if $a_0 = 0$ while $a_1$, $b_1$ are not both zero, the curve has at the origin a single tangent; the origin $O$ is therefore called a ***simple***, or ***ordinary***, ***point*** of the curve. In other words, *if the lowest terms in the equation* (1) *of an algebraic curve are of the first degree, the origin is a simple point of the curve, and the equation of the tangent at the origin is obtained by equating to zero the terms of the first degree.*

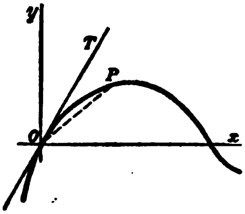

FIG. 90

## 170. Double Point.

The condition $a_1\cos\phi + b_1\sin\phi = 0$ necessary for two zero roots is also satisfied if $a_1 = 0$ and $b_1 = 0$; indeed, it is then satisfied whatever the value of $\phi$. Hence, if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$, the equation (2) has at least two zero roots for any value of $\phi$. If in this case the terms of the second degree in (1) do not all vanish, the curve is said to have a ***double point*** at the origin. Thus, *the origin is a double point if, and only if, the lowest terms in the equation* (1) *are of the second degree.*

## 171. Tangents at a Double Point.

The equation (2) will have at least three of its roots equal to zero if we have $a_0 = 0$, $a_1 = 0$, $b_1 = 0$ and

$$a_2\cos^2\phi + b_2\cos\phi\sin\phi + c_2\sin^2\phi = 0.$$

If $a_2$, $b_2$, $c_2$ are not all zero, we can find two angles satisfying this equation which may be real and different, or real and equal, or imaginary. The lines drawn at these angles (if real) through the origin are the *tangents at the double point.*

Multiplying the last equation by $r^2$ and reintroducing cartesian coordinates we obtain for these tangents the equation

$$a_2x^2 + b_2xy + c_2y^2 = 0. \tag{4}$$

Thus, *if the lowest terms in the equation* (1) *are of the second degree, the origin is a double point, and these terms of the second degree equated to zero represent the tangents at the origin.*

**172. Types of Double Point.** (*a*) If the two lines (4) are real and different, the double point is called a ***node*** or *crunode;* the curve then has two branches passing through the origin, each with a different tangent (Fig. 91).

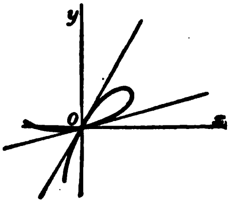

FIG. 91

(*b*) If the lines (4) are coincident, *i.e.* if $a_2x^2 + b_2xy + c_2y^2$ is a complete square, the double point is called a ***cusp***, or *spinode;* the curve then has ordinarily two real branches tangent to one and the same line at the origin (Fig. 92 represents the most simple case).

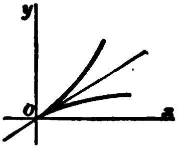

FIG. 92

(*c*) If the lines (4) are imaginary, the double point is called an ***isolated point***, or an *acnode;* in this case, while the coordinates 0, 0 of the origin satisfy the equation of the curve, there exists about the origin a region containing no other point of the curve, so that no tangents can be drawn through the origin (Fig. 93).

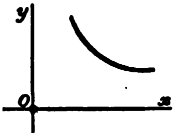

FIG. 93

It should be observed that, for curves of a degree above the third, the origin in case (*b*) may be an isolated point; this will be revealed by investigating the higher terms (viz. those above the second degree).

**173. Multiple Points.** It is readily seen how the reasoning of the last articles can be continued although the investigation of higher multiple points would require further discussion. The result is this: *If in the equation of an algebraic curve, when rationalized and cleared of fractions, the lowest terms are of degree k, the origin is a k-tuple point of the curve, and the tangents at this point are given by the terms of degree k, equated to zero.*

To investigate whether any given point $(x_1, y_1)$ of an algebraic curve is simple or multiple it is only necessary to transfer the origin to the point, by replacing $x$ by $x + x_1$ and $y$ by $y + y_1$, and then to apply this rule.

### EXERCISES

**1.** Determine the nature of the origin and sketch the curves:

(*a*) $y = x^2 - 2x$. (*b*) $x^2 = 4y - y^2$. (*c*) $(x + a)(y + a) = a^2$.
(*d*) $y^2 = x^2(4 - x)$. (*e*) $y^2 = x^3$. (*f*) $x^2 + y^2 = x^3$.
(*g*) $y^2 = x^2 + x^3$. (*h*) $x^3 - 3axy + y^3 = 0$. (*i*) $x^4 - y^4 + 6xy^2 = 0$.

**2.** Determine the nature of the origin and sketch the curve $(y - x^2)^2 = x^n$, for: (*a*) $n = 1$. (*b*) $n = 2$. (*c*) $n = 3$. (*d*) $n = 4$.

**3.** Locate the multiple points, determine their nature, and sketch the curves:

(*a*) $y^2 = x(x + 3)^2$. (*b*) $(y - 3)^2 = x^2$. (*c*) $(y + 1)^2 = (x - 3)^3$.
(*d*) $y^3 = (x + 1)(x - 1)^2$.

**4.** Sketch the curve $y^2 = (x - a)(x - b)(x - c)$ and discuss the multiple points when:

(*a*) $0 < a < b < c$. (*b*) $0 < a < b = c$. (*c*) $0 < a = b < c$. (*d*) $0 < a = b = c$.

## PART II. SPECIAL CURVES

**174. Conchoid.** *A fixed point $O$ and a fixed line $l$, at the distance $a$ from $O$, being given, the radius vector $OQ$, drawn from $O$ to every point $Q$ of $l$, is produced by a segment $QP = b$ of constant length;* the locus of $P$ is called the ***conchoid of Nicomedes.***

For $O$ as pole and the perpendicular to $l$ as polar axis the equation of $l$ is

$$r_1 = a/\cos\phi;$$

hence that of the conchoid is

$$r = \frac{a}{\cos\phi} + b.$$

If the segment $QP$ be laid off in the opposite sense, we obtain the curve

$$r = \frac{a}{\cos\phi} - b,$$

which is also called a conchoid. Indeed, these two curves are often regarded as merely two branches of the same curve. Transforming to cartesian coordinates and rationalizing, we find the equation

$$(x-a)^2(x^2+y^2) = b^2x^2,$$

which represents both branches. Sketch the curve, say for $b = 2a$, and for $b = a/2$, and determine the nature of the origin.

**175. Limaçon.** *If the line $l$ be replaced by a circle and the fixed point $O$ be taken on the circle,* the locus of $P$ is called ***Pascal's limaçon.***

For $O$ as pole and the diameter of the circle as polar axis the equation of the circle, of radius $a$, is $r_1 = 2a\cos\phi$; hence that of the limaçon is:

$$r = 2a\cos\phi + b.$$

If $b = 2a$, the curve is called the ***cardioid;*** the equation then becomes

$$r = 4a\cos^2 \tfrac{1}{2}\phi.$$

Sketch the limaçons for $b = 3a$, $2a$, $a$; transform to cartesian coordinates and determine the character of the origin.

**176. Cissoid.** *$OO' = a$ being a diameter of a circle, let any radius vector drawn from $O$ meet the circle and its tangent at $O'$ at the points $Q$, $D$, respectively;* if on this radius vector we lay off $OR = QD$, the locus of $R$ is called the ***cissoid of Diocles.***

With $O$ as pole and $OO'$ as polar axis, we have

$$OD = a/\cos\phi,\ OQ = a\cos\phi;$$

the equation is therefore

$$r = a\left(\frac{1}{\cos\phi} - \cos\phi\right) = a\frac{\sin^2\phi}{\cos\phi},$$

or in cartesian coordinates

$$y^2 = \frac{x^3}{a - x}.$$

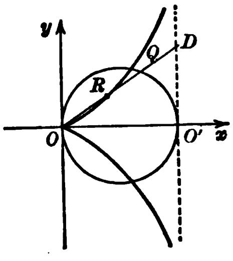

FIG. 94

If instead of taking the *difference* of the radii vectores of the circle and its tangent we take their *sum*, we obtain the so-called *companion of the cissoid,*

$$r = a(\cos\phi + \sec\phi),$$

*i.e.*

$$y^2 = x^2\frac{2a - x}{x - a}.$$

Sketch this curve.

**177. Versiera.** With the data of § 176, *let us draw through $Q$ a parallel to the tangent, through $D$ a parallel to the diameter;* the locus of the point of intersection $P$ of these parallels is called the ***versiera*** (wrongly called the "witch of Agnesi").

We have evidently with $O$ as origin and $OO'$ as axis $Ox$:

$$x = a\cos^2\phi, \qquad y = a\tan\phi,$$

whence eliminating $\phi$:

$$x = \frac{a^3}{y^2 + a^2}.$$

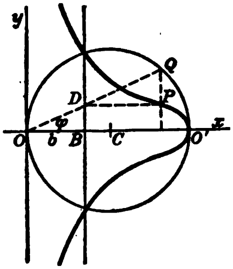

FIG. 95

If we replace the tangent at $O'$ by any perpendicular to $OO'$ (Fig. 95), at the distance $b$ from $O$, we obtain the curve

$$x = a\cos^2\phi, \qquad y = b\tan\phi,$$

*i.e.*

$$x = \frac{ab^2}{y^2 + b^2},$$

which reduces to the versiera for $b = a$.

Sketch the versiera, and the last curve for $b = \frac{1}{2}a$.

**178. Cassinian Ovals. Lemniscate.** Two fixed points $F_1$, $F_2$ being given it is known that the locus of a point $P$ is:

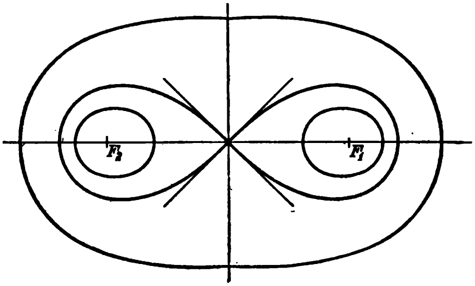

FIG. 96

(*a*) a *circle* if $F_1P/F_2P =$ const. (Ex. 7, p. 54);
(*b*) an *ellipse* if $F_1P + F_2P =$ const. (§ 114);
(*c*) a *hyperbola* if $F_1P - F_2P =$ const. (§ 119).

The locus is called a ***Cassinian oval*** if $F_1P \cdot F_2P =$ const. If we put $F_1F_2 = 2a$, the equation, referred to the midpoint $O$ between $F_1$ and $F_2$ as origin and $OF_2$ as axis $Ox$, is

$$[(x+a)^2+y^2]\,[(x-a)^2+y^2]=k^2.$$

In the particular case when $k=a^2$ the curve passes through the origin and is called a ***lemniscate.*** The equation then reduces to the form

$$(x^2+y^2)^2=2\,a^2(x^2-y^2),$$

which becomes in polar coordinates $r^2=2a^2\cos 2\phi$.

Trace the lemniscate from the last equation.

**179. Cycloid.** The ***common cycloid*** is *the path described by any point P of a circle rolling over a straight line* (Fig. 97).

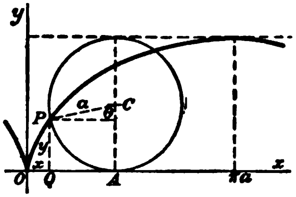

FIG. 97

If $A$ be the point of contact of the rolling circle in any position, $O$ the point of the given line that coincided with the point $P$ of the circle when $P$ was point of contact, it is clear that the length $OA$ must equal the arc $AP=a\theta$, where $a$ is the radius of the circle, and $\theta = \measuredangle ACP$, the angle through which the circle has turned since $P$ was at $O$. The figure then shows that, with $O$ as origin and $OA$ as axis $Ox$:

$$x=OQ=a\theta-a\sin\theta,\quad y=a-a\cos\theta.$$

These are the *parameter equations* of the cycloid. The curve has an infinite number of equal arches, each with an axis of symmetry (in Fig. 97, the line $x=\pi a$) and with a cusp at each end. Write down the cartesian equation.

**180. Trochoid.** The path described by any point $P$ rigidly connected with the rolling circle is called a ***trochoid.*** If the distance of $P$ from the center $C$ of the circle is $b$, the equations of the trochoid are $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$.

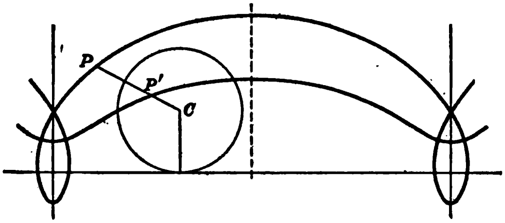

FIG. 98. — The Trochoids

Draw the trochoid for $b = \frac{1}{8} a$ and for $b = \frac{4}{8} a$.

**181. Epicycloid.** The path described by any point $P$ of a circle rolling on the outside of a fixed circle is called an ***epicycloid*** (Fig. 99).

Let $O$ be the center, $b$ the radius, of the fixed circle, $C$ the center, $a$ the radius, of the rolling circle; and let $A_0$ be that point of the fixed circle at which the describing point $P$ is the point of contact. Put $A_0OA = \phi$, $ACP = \theta$. As the arcs $AA_0$ and $AP$ are equal, we have $b\phi = a\theta$.

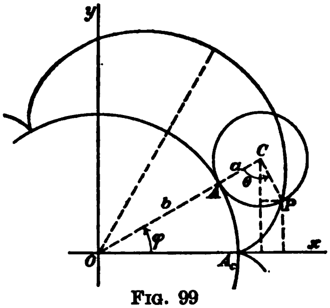

FIG. 99

With $O$ as origin and $OA_0$ as axis of $x$ we have

$$x = (a + b) \cos \phi + a \sin [\theta - (\tfrac{1}{2}\pi - \phi)],$$

$$y = (a + b) \sin \phi - a \cos [\theta - (\tfrac{1}{2}\pi - \phi)],$$

*i.e.*

$$x = (a + b) \cos \phi - a \cos \frac{a + b}{a} \phi,$$

$$y = (a + b) \sin \phi - a \sin \frac{a + b}{a} \phi.$$

**182. Hypocycloid.** If the circle rolls on the inside of the fixed circle, the path of any point of the rolling circle is called a ***hypocycloid.*** The equations are obtained in the same way; they differ from those of the epicycloid (§ 181) merely in having $a$ replaced by $-a$. Write down these equations.

Show that: ($a$) for $b=2a$ the hypocycloid reduces to a straight line, and illustrate this graphically; ($b$) for $b=4a$ the curve, called the *four-cusped hypocycloid,* has the equations

$$x=3a\cos\phi+a\cos 3\phi=a\cos^3\phi,$$
$$y=3a\sin\phi-a\sin 3\phi=a\sin^3\phi,$$

whence

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

### EXERCISES

**1.** Sketch the following curves: ($a$) Spiral of Archimedes $r=a\phi$; ($b$) Hyperbolic spiral $r\phi=a$; ($c$) Lituus $r^2\phi=a^2$.

**2.** Sketch the following curves: ($a$) $r=a\sin\phi$; ($b$) $r=a\cos\phi$; ($c$) $r=a\sin 2\phi$; ($d$) $r=a\cos 2\phi$; ($e$) $r=a\cos 3\phi$; ($f$) $r=a\sin 3\phi$; ($g$) $r=a\cos 4\phi$; ($h$) $r=a\sin 4\phi$.

**3.** Sketch with respect to the same axes the Cassinian ovals (§ 178) for $a=1$ and $k=2, 1.5, 1.1, 1, .75, .5, 0$.

**4.** Let two perpendicular lines $AB$ and $CD$ intersect at $O$. Through a fixed point $Q$ of $AB$ draw any line intersecting $CD$ at $R$. On this line lay off in both directions from $R$ segments $RP$ of length $OR$. The locus of $P$ is called the *strophoid.* Find the equation and sketch the curve.

**5.** Show that the lemniscate (§ 178) is the inverse curve of an equilateral hyperbola with respect to a circle about its center.

**6.** Show that the strophoid (Ex. 4) is the curve inverse to an equilateral hyperbola with respect to a circle about a vertex with radius equal to the transverse axis.

**7.** Show that the cissoid (§ 176) is the curve inverse to a parabola with respect to a circle about its vertex.

**8.** Find the curve inverse to the cardioid (§ 175) with respect to a circle about the origin.

9. Transform the equation $a(x^2 + y^2) = x^3$ to polar coordinates, indicate a geometrical construction, and draw the curve.

10. A tangent to a circle of radius $2a$ about the origin intersects the axes at $T$ and $T'$. Find and sketch the locus of the midpoint $P$ of $TT'$.

11. From any point $Q$ of the line $x = a$ draw a line parallel to the axis $Ox$ intersecting the axis $Oy$ at $C$. Find and sketch the locus of the foot of the perpendicular from $C$ on $OQ$.

12. The center of a circle of radius $a$ moves along the axis $Ox$. Find and sketch the locus of the intersections of this circle with lines joining the origin to its highest point.

13. The center of a circle of radius $a$ moves along the axis $Ox$. Find and sketch the locus of its points of contact with the lines through the origin.

**183. The Sine Curve.** The simple ***sine curve***, $y = \sin x$, is best constructed by means of an auxiliary circle of radius one. In Fig. 100, $OQ$ is made equal to the length of the arc $OA = x$; the ordinate at $Q$ is then equal to the ordinate $BA$ of the circle.

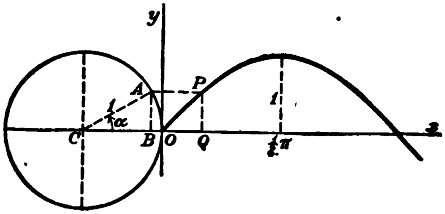

FIG. 100

Construct one whole *period* of the sine curve, *i.e.* the portion corresponding to the whole circumference of the auxiliary circle; the width $2\pi$ of this portion is called the period of the function $\sin x$.

The simple ***cosine curve***, $y = \cos x$, is the same as the sine curve except that the origin is taken at the point $(\frac{1}{2}\pi, 0)$.

The simple ***tangent curve***, $y = \tan x$, is derived like the sine curve from a unit circle. Its *period* is $\pi$.

**184. The Inverse Trigonometric Curves.** The equation $y = \sin x$ can also be written in the form

$$x = \sin^{-1} y, \quad \text{or } x = \text{arc} \sin y.$$

The curve represented by this equation is of course the same as that represented by the equation $y = \sin x$.

But if $x$ and $y$ be interchanged, the resulting equation

$$x = \sin y, \text{ or } y = \sin^{-1} x,\ y = \text{arc} \sin x,$$

represents the curve obtained from the simple sine curve by reflection in the line $y = x$ (§ 70).

Notice that the *trigonometric functions* $\sin x$, $\cos x$, $\tan x$, etc., are *one-valued, i.e.* to every value of $x$ belongs only one value of the function, while the *inverse trigonometric functions* $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc., are *many-valued;* indeed, to every value of $x$, at least in a certain interval, belongs an infinite number of values of the function.

**185. Transcendental Curves.** The trigonometric and inverse trigonometric curves, as well as, in general, the cycloids and trochoids, are ***transcendental curves***, so called because the relation between the cartesian coordinates $x$, $y$ cannot be expressed in finite form (*i.e.* without using infinite series) by means of the *algebraic* operations of addition, subtraction, multiplication, division, and raising to a power with a constant exponent.

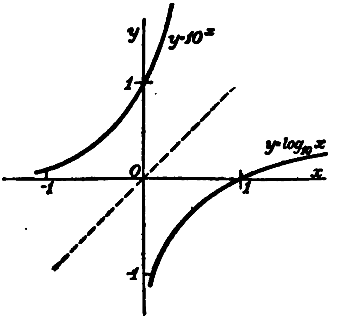

FIG. 101

**186. Logarithmic and Exponential Curves.** Another very important transcendental curve is the ***exponential curve***

$$y = a^x,$$

and its inverse, the ***logarithmic curve***

$$y = \log_a x,$$

where $a$ is any positive constant (Fig. 101). A full discussion of these curves can only be given in the calculus.

### EXERCISES

1. From a table of trigonometric functions, plot the curve $y = \sin x$.

2. Plot the curve $y = \sin x$ geometrically, as in § 183.

3. Plot the curve $y = \cos x$ (*a*) from a table; (*b*) by a geometric construction similar to that of § 183.

4. Plot the curve $y = \tan x$ from a table.

5. Plot each of the curves

(*a*) $y = \sin 2x$. (*b*) $y = 2 \cos 3x$. (*c*) $y = 3 \tan (x/2)$.
(*d*) $y = \sec x$. (*e*) $y = \cot 2x$. (*f*) $y = 2 \tan 4x$.

6. Plot each of the curves

(*a*) $y = \sin^{-1} x$. (*b*) $y = \cos^{-1} x$. (*c*) $y = \tan^{-1} x$.

7. By adding the ordinates of the two curves $y = \sin x$ and $y = \cos x$, construct the graph of $y = \sin x + \cos x$.

8. Draw each of the curves

(*a*) $y = \sin x + 2 \cos x$. (*c*) $y = \sec x + \tan x$.
(*b*) $y = 2 \sin x + \cos(x/2)$. (*d*) $y = \sin x + 2 \sin 2x + 3 \sin 3x$.

9. The equation $x = \sin t$, where $t$ means the time and $x$ means the distance of a body from its central position, represents a *Simple Harmonic Motion*. From the graph, describe the nature of the motion.

10. From a table of logarithms of numbers, draw the curve $y = \log_{10} x$.

11. By multiplying the ordinates of the curve of Ex. 10 by 3, construct the curve $y = 3 \log_{10} x$.

12. From the figure of Ex. 10, construct the curve $y = 10^x$ by reflection of the curve of Ex. 10 in the line $y = x$.

13. Draw the curve $y = \frac{1}{2} \log_{10} x$ by the process of Ex. 11. Show that it represents the equation $y = \log_{100} x$, since

$$y = \log_{100} x = \log_{100} 10 \times \log_{10} x = \tfrac{1}{2} \log_{10} x.$$

## PART III. EMPIRICAL EQUATIONS

**187. Empirical Formulas.** In scientific studies, the relations between quantities are usually not known in advance, but are to be found, if possible, from pairs of numerical values of the quantities discovered by experiment.

Simple cases of this kind have already been given in §§ 15, 29. In particular, the values of $a$ and $b$ in formulas of the type $y = a + bx$ were found from two pairs of values of $x$ and $y$. Compare also § 34.

Likewise, if two quantities $y$ and $x$ are known to be connected by a relation of the form $y = a + bx + cx^2$, the values of $a, b, c$ can be found from any *three* pairs of values of $x$ and $y$. For, if any pair of values of $x$ and $y$ are substituted for $x$ and $y$ in this equation, we obtain a linear equation for $a$, $b$, and $c$. Three such equations usually determine $a$, $b$, and $c$.

In general the coefficients $a, b, c, \cdots, l$ in an equation of the type

$$y = a + bx + cx^2 + \cdots + lx^n$$

can be found from any $n + 1$ pairs of values of $x$ and $y$.

**188. Approximate Nature of Results.** Since the measurements made in any experiment are liable to at least small errors, it is not to be expected that the calculated values of such coefficients as $a, b, c, \cdots$ of § 187 will be absolutely accurate, nor that the points that represent the pairs of values of $x$ and $y$ will all lie absolutely on the curve represented by the final formula.

To increase the accuracy, a large number of pairs of values of $x$ and $y$ are usually measured experimentally, and various pairs are used to determine such constants as $a, b, c, \cdots$ of § 187. The *average* of all the computed values of any one such constant is often taken as a fair approximation to its true value.

### 189. Illustrative Examples.

EXAMPLE 1. A wire under tension is found by experiment to stretch an amount $l$, in thousandths of an inch, under a tension $T$, in pounds, as follows: —

| | | | | | |
|---|---|---|---|---|---|
| $T$ in pounds . . . . . . | 10 | 15 | 20 | 25 | 30 |
| $l$ in thousandths of an inch . | 8 | 12.5 | 15.5 | 20 | 23 |

Find a relation of the form $l = kT$ (*Hooke's Law*) which approximately represents these results.

First plot the given data on squared paper, as in the adjoining figure.

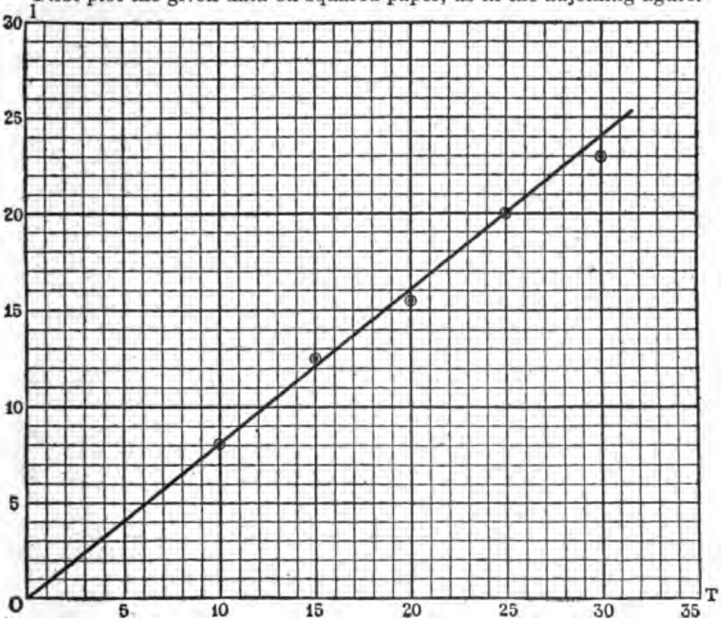

FIG. 102

Substituting $l = 8$, $T = 10$ in $l = kT$, we find $k = .8$. From $l = 12.5$, $T = 15$, we find $k = .833$. Likewise, the other pairs of values of $l$ and $T$ give, respectively, $k = .775$, $k = .8$, $k = .767$. The average of all these values of $k$ is $k = .795$; hence we may write, approximately,

$$l = .795\ T.$$

This equation is represented by the line in Fig. 102; this line does not pass through even one of the given points, but it is a fair compromise between all of them, in view of the fact that each of them is itself probably slightly inaccurate.

**Example 2.** In an experiment with a Weston Differential Pulley Block, the effort $E$, in pounds, required to raise a load $W$, in pounds, was found to be as follows:

| $W$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $E$ | $3\frac{1}{4}$ | $4\frac{7}{8}$ | $6\frac{1}{4}$ | $7\frac{1}{2}$ | 9 | $10\frac{1}{2}$ | $12\frac{1}{4}$ | $13\frac{3}{4}$ | 15 | $16\frac{1}{2}$ |

Find a relation of the form $E = aW + b$ that approximately agrees with these data. [Gibson]

These values may be plotted in the usual manner on squared paper. They will be found to lie very nearly on a straight line. If $E$ is plotted vertically, $b$ is the intercept on the vertical axis, and $a$ is the slope of the line; both can be measured directly in the figure.

To determine $a$ and $b$ more exactly, we may take various points that lie nearly on the line. Thus $(E = 6\frac{1}{4},\ W = 30)$ and $(E = 16\frac{1}{2},\ W = 100)$ lie nearly on a line that passes close to all the points. Substituting in the equation $E = aW + b$ we obtain

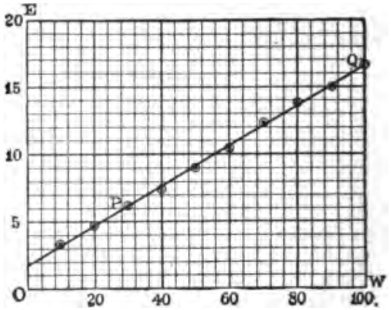

Fig. 103

$$6\tfrac{1}{4} = 30a + b, \qquad 16\tfrac{1}{2} = 100a + b,$$

whence $a = 0.146$, $b = 1.86$. Hence we may take

$$E = 0.146\,W + 1.86,$$

approximately. Other pairs of values of $E$ and $W$ may be used in like manner to find values for $a$ and $b$, and all the values of each quantity may be averaged.

Example 3. If $\theta$ denotes the melting point (Centigrade) of an alloy of lead and zinc containing $x$ per cent of lead, it is found that

| | | | | | | |
|---|---|---|---|---|---|---|
| $x = \%$ lead . . . . . . . . | 40 | 50 | 60 | 70 | 80 | 90 |
| $\theta =$ melting point . . . . | 186° | 205° | 226° | 250° | 276° | 304° |

Find a relation of the form $\theta = a + bx + cx^2$ that approximately expresses these facts. [Saxelby]

Taking any three pairs of values, say (40, 186), (70, 250), (90, 304), and substituting in $\theta = a + bx + cx^2$ we find

$$186 = a + 40\,b + 1600\,c,$$
$$250 = a + 70\,b + 4900\,c,$$
$$304 = a + 90\,b + 8100\,c,$$

whence $a = 132$, $b = .92$, $c = .0011$, approximately; whence

$$\theta = 132 + .92\,x + .0011\,x^2.$$

Other sets of three pairs of values of $x$ and $y$ may be used in a similar manner to determine $a$, $b$, $c$; and the resulting values averaged, as above.

### EXERCISES

1. In experiments on an iron rod, the amount of elongation $l$ (in thousandths of an inch) and the stretching force $p$ (in thousands of pounds) were found to be $(p = 10,\ l = 8)$, $(p = 20,\ l = 15)$, $(p = 40,\ l = 31)$. Find a formula of the type $l = k \cdot p$ which approximately expresses these data. *Ans.* $k = .775$.

2. The values 1 in. = 2.5 cm. and 1 ft. = 30.5 cm. are frequently quoted, but they do not agree precisely. The number of centimeters, $c$, in $i$ inches is surely given by a formula of the type $c = ki$. Find $k$ approximately from the preceding data.

3. The readings of a standard gas-meter $S$ and those of a meter $T$ being tested on the same pipe-line were found to be $(S=3000,\ T=0)$, $(S=3510,\ T = 500)$, $(S = 4022,\ T = 1000)$. Find a formula of the type $T = aS + b$ which approximately represents these data.

4. An alloy of tin and lead containing $x$ per cent of lead melts at the temperature $\theta$ (Fahrenheit) given by the values $(x = 25\%,\ \theta = 482°)$, $(x = 50\%,\ \theta = 370°)$, $(x = 75\%,\ \theta = 356°)$. Determine a formula of the type $\theta = a + bx + cx^2$ which approximately represents these values.

5. The temperatures $\theta$ (Centigrade) at a depth $d$ (feet) below the surface of the earth in a mine were found to be $d = 100$, $\theta = 15.7°$; $d = 200$, $\theta = 16.5$; $d = 300$, $\theta = 17.4$. Find a relation of the form $\theta = a + bd$ between $\theta$ and $d$.

6. Determine a line that passes reasonably near each of the three points (2, 4), (6, 7), (10, 9). Determine a quadratic expression $y = a + bx + cx^2$ that represents a parabola through the same three points.

7. Determine a parabola whose equation is of the form $y = a + bx + cx^2$ that passes through each of the points (0, 2.5), (1.5, 1.5), and (3.0, 2.8). Are the values of $a$, $b$, $c$ changed materially if the point (2.0, 1.7) is substituted for the point (1.5, 1.5)?

8. If the curve $y = \sin x$ is drawn with one unit space on the $x$-axis representing 60°, the points (0, 0), $(\frac{1}{2}, \frac{1}{2})$, $(1\frac{1}{2}, 1)$ lie on the curve. Find a parabola of the form $y = a + bx + cx^2$ through these three points, and draw the two curves on the same sheet of paper to compare them.

**190. Substitutions.** It is particularly easy to test whether points that are given by an experiment really lie on a straight line; that is, whether the quantities measured satisfy an equation of the form $y = a + bx$. This is done by means of a transparent ruler or a stretched rubber band.

For this reason, if it is suspected that two quantities $x$ and $y$ satisfy an equation of the form

$$y = a + bx^2,$$

it is advantageous to substitute a new letter, say $u$, for $x^2$:

$$u = x^2, \quad y = a + bu$$

and then plot the values of $y$ and $u$. If the new figure does agree reasonably well with some straight line, it is easy to find $a$ and $b$, as in § 189.

Likewise, if it is suspected that two quantities $x$ and $y$ are connected by a relation of the form

$$y = a + b \cdot \frac{1}{x} \text{ or } xy = ax + b,$$

it is advantageous to make the substitution $u = 1/x$.

Other substitutions of the same general nature are often useful.

*In any case, the given values of $x$ and $y$ should be plotted first unchanged, in order to see what substitution might be useful.*

**191. Illustrative Example.** If a body slides down an inclined plane, the distance $s$ that it moves is connected with the time $t$ after it starts by an equation of the form $s = kt^2$. Find a value of $k$ that agrees reasonably with the following data:

| | | | | | |
|---|---|---|---|---|---|
| $s$, in feet . . . . . . . | 2.6 | 10.1 | 23.0 | 40.8 | 63.7 |
| $t$, in seconds . . . . . | 1 | 2 | 3 | 4 | 5 |

In this case, it is not necessary to plot the values of $s$ and $t$ themselves, because the nature of the equation, $s = kt^2$, is known from physics.

Hence we make the substitution $t^2 = u$, and write down the supplementary table:

| | | | | | |
|---|---|---|---|---|---|
| $s$, in feet . . . . . . . | 2.6 | 10.1 | 23.0 | 40.8 | 63.7 |
| $u$ (or $t^2$) . . . . . . . | 1 | 4 | 9 | 16 | 25 |

These values will be found to give points very nearly on a straight line whose equation is of the form $s = ku$. To find $k$, we divide each value of $s$ by the corresponding value of $u$; this gives several values of $k$:

| | | | | | |
|---|---|---|---|---|---|
| $k$ | 2.6 | 2.525 | 2.556 | 2.55 | 2.548 |

The *average* of these values of $k$ is approximately 2.556; hence we may write $s = 2.556\,u$, or $s = 2.556\,t^2$.

### EXERCISES

**1.** Find a formula of the type $u = kv^2$ that represents approximately the following values:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $u$ | 3.9 | 15.1 | 34.5 | 61.2 | 95.5 | 137.7 | 187.4 |
| $v$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

**2.** A body starts from rest and moves $s$ feet in $t$ seconds according to the following measured values:

| | | | | | | |
|---|---|---|---|---|---|---|
| $s$, in feet . . . . . . | 3.1 | 13.0 | 30.6 | 50.1 | 79.5 | 116.4 |
| $t$, in seconds . . . . . | .5 | 1 | 1.5 | 2 | 2.5 | 3 |

Find approximately the relation between $s$ and $t$.

**3.** The pressure $p$, measured in centimeters of mercury, and the volume $v$, measured in cubic centimeters, of a gas kept at constant temperature, were found to be:

| | | | | | |
|---|---|---|---|---|---|
| $v$ | 145 | 155 | 165 | 178 | 191 |
| $p$ | 117.2 | 109.4 | 102.4 | 95.0 | 88.6 |

Substitute $u$ for $1/v$, compute the values of $u$, and determine a relation of the form $p = ku$; that is, $p = k/v$.

**4.** Determine a relation of the form $y = a + bx^2$ that approximately represents the values:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $y$ | 14.1 | 25.2 | 44.7 | 71.4 | 105.6 | 147.9 | 197.7 |

**192. Logarithmic Plotting.** In case the quantities $y$ and $x$ are connected by a relation of the form

$$y = kx^n,$$

it is advantageous to take logarithms (to the base 10) on both sides:

$$\log y = \log kx^n = \log k + n \log x,$$

and then substitute new letters for $\log x$ and $\log y$:

$$u = \log x, \qquad v = \log y.$$

For, if we do so, the equation becomes

$$v = l + nu,$$

where $l = \log k$.

If the values of $x$ and $y$ are given by an experiment, and if $u = \log x$ and $v = \log y$ are computed, the values of $u$ and $v$ should correspond to points that lie on a straight line, and the values of $l$ and $n$ can be found as in § 189. The value of $k$ may be found from that of $l$, since $\log k = l$.

EXAMPLE 1. The amount of water $A$, in cu. ft., that will flow per minute through 100 feet of pipe of diameter $d$, in inches, with an initial pressure of 50 lb. per sq. in., is as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| $d$ | 1 | 1.5 | 2 | 3 | 4 | 6 |
| $A$ | 4.88 | 13.43 | 27.50 | 75.13 | 152.51 | 409.54 |

Find a relation between $A$ and $d$.

Let $u = \log d$, $v = \log A$; then the values of $u$ and $v$ are

| | | | | | | |
|---|---|---|---|---|---|---|
| $u = \log d$ . . . | 0.000 | 0.176 | 0.301 | 0.477 | 0.602 | 0.778 |
| $v = \log A$ . . . | 0.688 | 1.128 | 1.439 | 1.876 | 2.183 | 2.612 |

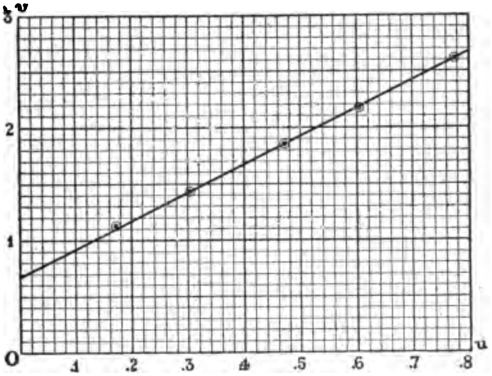

FIG. 104

These values give points in the $(u, v)$ plane that are very nearly on a straight line; hence we may write, approximately,

$$v = a + bu,$$

where $a$ and $b$ can be determined directly by measurement in the figure, or as in § 189. If we take the first and last pairs of values of $u$ and $v$, we find

$$.688 = a + 0,$$
$$2.612 = a + .778\,b.$$

Solving these equations, we find approximately, $a = .688$, $b = 2.473$, and we may write

$$v = .688 + 2.473\,u \quad \text{or} \quad \log A = .688 + 2.473 \log d.$$

Since

$$.688 = \log 4.88,$$

the last equation may be written in the form

$$\log A = \log 4.88 + 2.473 \log d$$
$$= \log(4.88\, d^{2.473})$$

whence

$$A = 4.88\, d^{2.473}.$$

Slightly different values of the constants may be found by using other pairs of values of $u$ and $v$.

**193. Logarithmic Paper.** Paper called logarithmic paper may be bought that is ruled in lines whose distances, horizontally and vertically, from one point $O$ (Fig. 105) are *proportional to the logarithms* of the numbers 1, 2, 3, etc.

Such paper may be used advantageously instead of actually looking up the logarithms in a table, as was done in § 192. For if the *given values* be plotted on this new paper, the resulting figure is identically the same as that obtained by plotting the *logarithms of the given values* on ordinary squared paper.

Example. A strong rubber band stretched under a pull of $p$ kg. shows an elongation of $E$ cm. The following values were found in an experiment:

| $p$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 6.0 | 7.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $E$ | 0.1 | 0.3 | 0.6 | 0.9 | 1.3 | 1.7 | 2.2 | 2.7 | 3.3 | 3.9 | 5.3 | 6.9 |

[Riggs]

If these values are plotted on logarithmic paper as in Fig. 105, it is evident that they lie reasonably near a straight line, such as that drawn.

By measurement in the figure, the slope of this line is found to be 1.6, approximately. Hence if $u = \log p$ and $v = \log E$, we have

$$v = l + 1.6\,u,$$

where $l$ is a constant not yet determined; whence

$$\log E = l + 1.6 \log p$$

or

$$E = kp^{1.6}$$

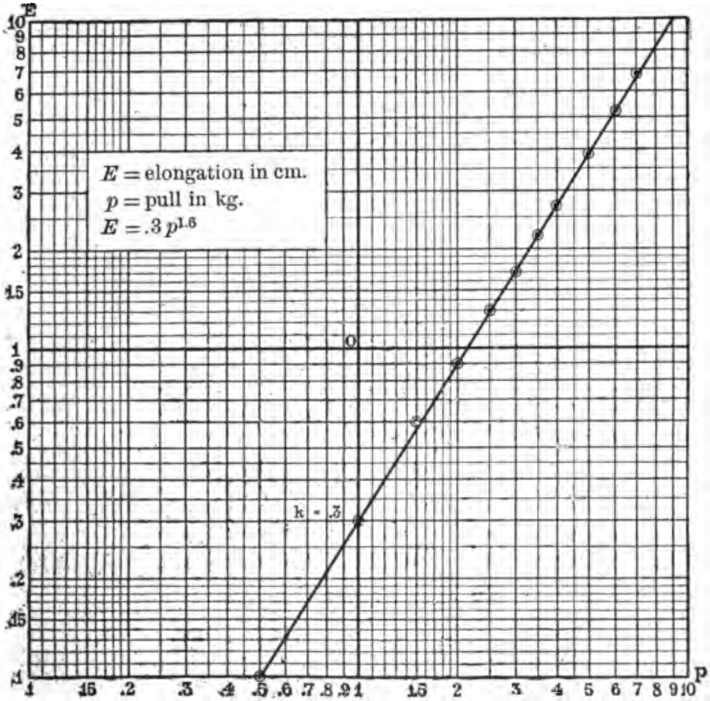

FIG. 105. — Elongation of a Rubber Band

where $l = \log k$. If $p = 1$, $E = k$; from the figure, if $p = 1$, $E = .3$; hence $k = .3$, and

$$E = .3\,p^{1.6}.$$

The use of logarithmic paper is however not at all essential; the same results may be obtained by the method of § 192.

### EXERCISES

**1.** In testing a gas engine corresponding values of the pressure $p$, measured in pounds per square foot, and the volume $v$, in cubic feet, were obtained as follows: $v = 7.14$, $p = 54.6$; 7.73, 50.7; 8.59, 45.9. Find the relation between $p$ and $v$ (use logarithmic plotting).

*Ans.* $p = 387.6\, v^{-.938}$, or $pv^{.938} = 387.6$.

**2.** Expansion or contraction of a gas is said to be adiabatic when no heat escapes or enters. Determine the adiabatic relation between pressure $p$ and volume $v$ (Ex. 1) for air from the following observed values: $p = 20.54$, $v = 6.27$; 25.79, 5.34; 54.25, 3.15.

*Ans.* $pv^{1.41} = 273.5$.

**3.** The intercollegiate track records for foot-races are as follows, where $d$ means the distance run, and $t$ means the record time:

| $d$ | 100 yd. | 220 yd. | 440 yd. | 880 yd. | 1 mi. | 2 mi. |
|---|---|---|---|---|---|---|
| $t$ | 0:09$\frac{4}{5}$ | 0:21$\frac{1}{5}$ | 0:48 | 1:54$\frac{4}{5}$ | 4:15$\frac{2}{5}$ | 9:24$\frac{2}{5}$ |

Plot the logarithms of these values on squared paper (or plot the given values themselves on logarithmic paper). Find a relation of the form $t = kd^n$. What should be the record time for a race of 1320 yd.?

[See Kennelly, *Popular Science Monthly*, Nov. 1908.]

**4.** Solve the Example of § 193 by the method of § 192.

**5.** Each of the following sets of quantities was found by experiment. Find in each case an equation connecting the two quantities, by §§ 192–193.

| (*a*) $v$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p$ | 137.4 | 62.6 | 39.6 | 28.6 | 22.6 |

| (*b*) $u$ | 12.9 | 17.1 | 23.1 | 28.5 | 3.0 |
|---|---|---|---|---|---|
| $v$ | 63.0 | 27.0 | 13.8 | 8.5 | 6.9 |

| (*c*) $\theta$ | 32° | 212° | 390° | 570° | 750° | 1100° |
|---|---|---|---|---|---|---|
| $c$ | 2.09 | 2.69 | 2.90 | 2.98 | 3.09 | 3.28 |

# SOLID ANALYTIC GEOMETRY

## CHAPTER X

### COORDINATES

**194. Location of a Point.** The position of a point in three-dimensional space can be assigned without ambiguity by giving its distances from three mutually rectangular planes, provided these distances are taken with proper signs according as the point lies on one or the other side of each plane.

The three planes, each perpendicular to the other two, are called the ***coordinate planes;*** their common point $O$ (Fig. 106) is called the ***origin.*** The three mutually rectangular lines $Ox$, $Oy$, $Oz$ in which the planes intersect are called the ***axes*** of coordinates; on each of them a positive sense is selected arbitrarily, by affixing the letter $x$, $y$, $z$, respectively.

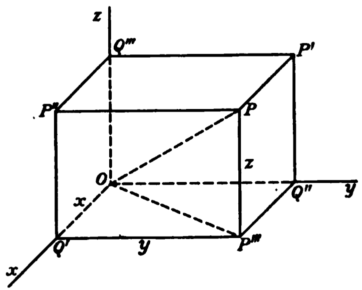

FIG. 106

The three coordinate planes, $Oyz$, $Ozx$, $Oxy$, divide the whole of space into eight compartments called ***octants.*** The first octant in which all three coordinates are positive is also called the ***coordinate trihedral.***

If $P'$, $P''$, $P'''$ are the projections of any point $P$ on the coordinate planes $Oyz$, $Ozx$, $Oxy$, respectively, then $P'P = x$, $P''P = y$, $P'''P = z$ are the ***rectangular cartesian coordinates*** of

$P$. If the planes through $P$ parallel to $Oyz$, $Ozx$, $Oxy$ intersect the axes $Ox$, $Oy$, $Oz$ in $Q'$, $Q''$, $Q'''$, the point $P$ is found from its coordinates $x$, $y$, $z$ by passing along the axis $Ox$ through the distance $OQ' = x$, parallel to $Oy$ through the distance $Q'P''' = y$, and parallel to $Oz$ through the distance $P'''P = z$, each of these distances being taken with the proper sense.

*Every point in space has three definite real numbers as coordinates;* conversely, *to every set of three real numbers corresponds one and only one point.*

Locate the points: $(2, 3, 4)$, $(-3, 2, 0)$, $(5, 0, -3)$, $(0, 0, 4)$, $(0, -6, 0)$, $(-5, -8, -2)$.

**195. Distance of a Point from the Origin.** For the distance $OP = r$ (Fig. 106) of the point $P(x, y, z)$ from the origin $O$ we have, since $OP$ is the diagonal of a rectangular parallelepiped with edges $OQ' = x$, $OQ'' = y$, $OQ''' = z$:

$$r = \sqrt{x^2 + y^2 + z^2}.$$

**196. Distance between two Points.** The distance between the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ can be found if the coordinates of the two points are given. For (Fig. 107), the planes through $P_1$ and those through $P_2$ parallel to the coordinate planes bound a rectangular parallelepiped with $P_1P_2 = d$ as diagonal; and as its edges are

$$P_1Q = x_2 - x_1, \quad P_1R = y_2 - y_1, \quad P_1S = z_2 - z_1,$$

we find

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

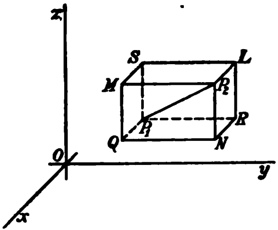

FIG. 107

**197. Oblique Axes.** The position of a point $P$ in space can also be determined with respect to three axes *not* at right angles. The coordinates of $P$ are the segments cut off on the axes by planes through $P$

parallel to the coordinate planes. In what follows, the axes are always assumed to be at right angles unless the contrary is definitely stated.

## EXERCISES

**1.** What are the coordinates of the origin? What can you say of the coordinates of a point on the axis $Ox$? on the axis $Oy$? on the axis $Oz$?

**2.** What can you say of the coordinates of a point that lies in the plane $Oxy$? in the plane $Oyz$? in the plane $Ozx$?

**3.** Where is a point situated when $x = 0$? when $z = 0$? when $x = y = 0$? when $y = z$? when $x = 2$? when $z = -3$? when $x = 1$, $y = 2$?

**4.** A rectangular parallelepiped lies in the first octant with three of its faces in the coordinate planes, its edges are of length $a$, $b$, $c$, respectively; what are the coordinates of the vertices?

**5.** Show that the points $(4, 3, 5)$, $(2, -1, 3)$, $(0, 1, 7)$ are the vertices of an equilateral triangle.

**6.** Show that the points $(-1, 1, 3)$, $(-2, -1, 4)$, $(0, 0, 5)$ lie on a sphere whose center is $(2, -3, 1)$. What is the radius of this sphere?

**7.** Show that the points $(6, 2, -5)$, $(2, -4, 7)$, $(4, -1, 1)$ lie on a straight line.

**8.** Show that the triangle whose vertices are $(a, b, c)$, $(b, c, a)$, $(c, a, b)$ is equilateral.

**9.** What are the coordinates of the projections of the point $(6, 3, -8)$ on the axes of coordinates? What are the distances of this point from the coordinate axes?

**10.** What is the length of the segment of a line whose projections on the coordinate axes are 5, 3, and 2?

**11.** What are the coordinates of the points which are symmetric to the point $(a, b, c)$ with respect to the coordinate planes? with respect to the axes? with respect to the origin?

**12.** Show that the sum of the squares of the four diagonals of a rectangular parallelepiped is equal to the sum of the squares of its edges.

**198. Projection.** The ***projection*** of a point on a plane or line is the foot of the perpendicular let fall from the point on the plane or line. The projection of a rectilinear segment $AB$ on a plane or line is the intercept $A'B'$ between the feet of the perpendiculars $AA'$, $BB'$ let fall from $A$, $B$ on the plane or line. If $\alpha$ is one of the two angles made by the segment with the plane or line, we have

$$A'B' = AB \cos \alpha.$$

In analytic geometry we have generally to project a ***vector***, *i.e.* a segment with a definite sense, on an *axis*, *i.e.* on a line with a definite sense (compare § 19). The angle $\alpha$ is then understood to be the angle between the positive senses of vector and axis (both being drawn from a common origin). The above formula then gives the projection with its proper sign.

Thus, the segment $OP$ (Fig. 106) from the origin to any point $P(x, y, z)$ can be regarded as a vector $OP$. Its projections on the axes of coordinates are the coordinates $x$, $y$, $z$ of $P$. These projections are also called the *rectangular components* of the vector $OP$, and $OP$ is called the *resultant* of the components $OQ'$, $OQ''$, $OQ'''$, or also of $OQ'$, $Q'P'''$, $P'''P$.

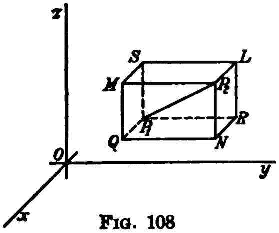

FIG. 108

Similarly, in Fig. 108, if $P_1P_2$ be regarded as a vector, the projections of this vector $P_1P_2$ on the axes of coordinates are the coordinate differences $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$. See § 203.

**199. Resultant.** The proposition of § 19 that *the sum of the projections of the sides of an open polygon on any axis is* *equal to the projection of the closing side on the same axis* and that of § 20 that *the projection of the resultant is equal to the sum of the projections of its components* are readily seen to hold in three dimensions as well as in the plane. Analytically these propositions follow by considering that whatever the points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $\cdots$ $P_n(x_n, y_n, z_n)$ in space, the sum of the projections of the vectors $P_1P_2, P_2P_3, \cdots P_{n-1}P_n$ on the axis $Ox$ is:

$$(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) = x_n - x_1,$$

where the right-hand member is the projection of the closing side or resultant $P_1P_n$ on $Ox$. Any line can of course be taken as axis $Ox$.

**200. Division Ratio.** *Two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ being given by their coordinates, the coordinates $x$, $y$, $z$ of any point $P$ of the line $P_1P_2$ can be found if the division ratio $P_1P/P_1P_2 = k$ is known in which the point $P$ divides the segment* $P_1P_2$ (Fig 109).

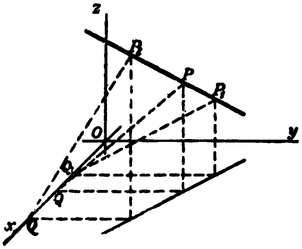

FIG. 109

Let $Q_1$, $Q$, $Q_2$ be the projections of $P_1$, $P$, $P_2$ on the axis $Ox$; as $Q$ divides $Q_1Q_2$ in the same ratio $k$ in which $P$ divides $P_1P_2$, we have as in § 3:

$$x = x_1 + k(x_2 - x_1).$$

Similarly we find by projecting on $Oy$, $Oz$:

$$y = y_1 + k(y_2 - y_1), \quad z = z_1 + k(z_2 - z_1).$$

If $k$ is positive, $P$ lies on the same side of $P_1$ as does $P_2$; if $k$ is negative, $P$ lies on the opposite side of $P_1$ (§ 3).

**201. Direction Cosines.** Instead of using the cartesian coordinates $x$, $y$, $z$ to locate a point $P$ (Fig. 110) we can also use its ***radius vector*** $r = OP$, *i.e.* the length of the vector drawn from the origin to the point, and its ***direction cosines***, *i.e.* the cosines of the angles $\alpha$, $\beta$, $\gamma$, made by the vector $OP$ with the axes $Ox$, $Oy$, $Oz$. We have evidently

$$x = r\cos\alpha,\quad y = r\cos\beta,\quad z = r\cos\gamma.$$

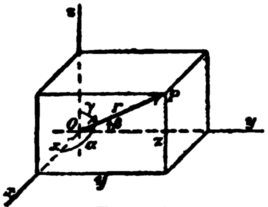

Fig. 110

As a line has two opposite senses we can take as *direction cosines of any line* parallel to $OP$ either $\cos\alpha$, $\cos\beta$, $\cos\gamma$, or $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

The direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of a vector $OP$ are often denoted briefly by the letters $l$, $m$, $n$, respectively, so that the coordinates of $P$ are

$$x = lr,\quad y = mr,\quad z = nr.$$

The direction cosines of any parallel line are then $l$, $m$, $n$ or $-l$, $-m$, $-n$.

**202. Pythagorean Relation.** *The sum of the squares of the direction cosines of any line is equal to one.*

For the equations of § 201 give upon squaring and adding, since $x^2 + y^2 + z^2 = r^2$:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

or

$$l^2 + m^2 + n^2 = 1;$$

and this still holds when $l$, $m$, $n$ are replaced by $-l$, $-m$, $-n$. Since this result is derived directly from the Pythagorean Theorem of geometry, it may be called the *Pythagorean Relation* between the direction cosines. Notice that $l$, $m$, $n$ can be regarded as the coordinates of the extremity of a vector of unit length drawn from the origin parallel to the line.

## EXERCISES

**1.** Find the length of the radius vector and its direction cosines for each of the following points: $(5, -3, 2)$; $(-3, -2, 1)$; $(-4, 0, 8)$.

**2.** The direction cosines of a line are proportional to 1, 2, 3; find their values.

**3.** A straight line makes an angle of 30° with the axis $Ox$ and an angle of 60° with the axis $Oy$; what is the third direction angle?

**4.** What is the direction of a line when $l = 0$? when $l = m = 0$?

**5.** What are the direction cosines of that line whose direction angles are equal?

**6.** What are the direction cosines of the line bisecting the angle between two intersecting lines whose direction cosines are $l$, $m$, $n$ and $l'$, $m'$, $n'$, respectively?

**7.** Find the direction cosines of the line which bisects the angle between the radii vectores of the points $(3, -4, 2)$ and $(-1, 2, 3)$.

**8.** Three vertices of a parallelogram are $(4, 3, -2)$, $(7, -1, 4)$, $(-2, 1, -4)$; find the coordinates of the fourth vertex (three solutions).

**9.** In what ratio is the line drawn from the point $(2, -5, 8)$ to the point $(4, 6, -2)$ divided by the plane $Ozx$? by the plane $Oxy$? At what points does this line pierce these coordinate planes?

**10.** In what ratio is the line drawn from the point $(0, 5, 0)$ to the point $(8, 0, 0)$ divided by the line in the plane $Oxy$ which bisects the angle between the axes?

**11.** Find the coordinates of the midpoint of the line joining the points $(4, -3, 8)$ and $(6, 5, -9)$. Find the points which trisect the same segment.

**12.** If we add to the segment joining the points $(4, 1, 2)$ and $(-2, 5, 7)$ a segment of twice its length in each direction, what are the coordinates of the end points?

**13.** Find the coordinates of the intersection of the medians of the triangle whose vertices are $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$.

**14.** Show that the lines joining the midpoints of the opposite edges of a tetrahedron intersect and are bisected by their common point.

**15.** Show that the projection of the radius vector of the point $P(x, y, z)$ on a line whose direction cosines are $l'$, $m'$, $n'$ is $l'x + m'y + n'z$.

**203. Projections. Components of a Vector.** If two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by their coordinates, the *projections of the vector*, $P_1P_2$ on the axes, or what amounts to the same, on parallels to the axes drawn through $P_1$ (Fig. 111), are evidently (§ 198):

$$P_1Q = x_2 - x_1, \quad P_1R = y_2 - y_1,$$
$$P_1S = z_2 - z_1.$$

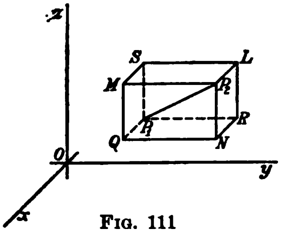

Fig. 111

These projections, or also the vectors $P_1Q$, $QN$, $NP_2$, are called the *rectangular components* of the vector $P_1P_2$, or its *components along the axes.*

If $d$ is the length of the segment $P_1P_2$, its direction cosines $l$, $m$, $n$ are, since $P_2Q$ is perpendicular to $P_1Q$, $P_2R$ to $P_1R$, $P_2S$ to $P_1S$:

$$l = \frac{x_2 - x_1}{d}, \quad m = \frac{y_2 - y_1}{d}, \quad n = \frac{z_2 - z_1}{d}.$$

These relations can also be written in the form:

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = d.$$

**204. Angle between Two Lines.** *If the directions of two lines are given by their direction cosines $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$, the angle $\psi$ between the two lines is given by the formula*

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

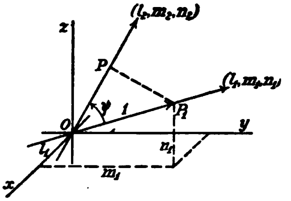

Fig. 112

For, drawing through the origin two lines of direction cosines $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ and taking on the former a vector $OP_1$ of unit length, the projection $OP$ of $OP_1$ on the other line is equal to the cosine of the required angle $\psi$. On the other hand, $OP_1$ has $l_1$, $m_1$, $n_1$ as components along the axes; hence, by § 199:

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

Two intersecting lines (or any two parallels to them) make two angles, say $\psi$ and $\pi - \psi$. But if the direction cosines of each line are given, a definite sense has been assigned to each line, and the angle between the lines is understood to be the angle between these senses.

**205. Conditions for Parallelism and for Perpendicularity.** If, in particular, the lines are parallel, we have either $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$, or $l_1 = -l_2$, $m_1 = -m_2$, $n_1 = -n_2$; hence in either case

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

This then is the ***condition of parallelism*** of two lines whose direction cosines are $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$.

If the lines are perpendicular, *i.e.* if $\psi = \frac{1}{2}\pi$, we have $\cos\psi = 0$; hence the ***condition of perpendicularity*** of two lines whose direction cosines are $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

**206.** The formula of § 204 gives

$$\sin^2\psi = 1 - \cos^2\psi = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2.$$

As (§ 202) $(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) = 1$, we can write this expression in the form

$$\sin^2\psi = \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 \end{vmatrix},$$

which can also be expressed as follows:

$$\sin^2\psi = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2.$$

The *direction* $(l, m, n)$ *perpendicular to two given different directions* $(l_1, m_1, n_1)$ and $(l_2, m_2, n_2)$ is found by solving the equations (§ 205)

$$l_1l + m_1m + n_1n = 0,$$
$$l_2l + m_2m + n_2n = 0,$$

whence
$$\frac{l}{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}} = \frac{m}{\begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}} = \frac{n}{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}}.$$

If we denote by $k$ the common value of these ratios, we have
$$l = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} k, \quad m = \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} k, \quad n = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} k;$$
substituting these values in the relation (§ 202) $l^2 + m^2 + n^2 = 1$, and observing the preceding value of $\sin\psi$, we find:
$$l = \pm \frac{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}}{\sin\psi}, \quad m = \pm \frac{\begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}}{\sin\psi}, \quad n = \pm \frac{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}}{\sin\psi},$$
where $\psi$ is the angle between the given directions.

**207.** Three directions $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, $(l_3, m_3, n_3)$ are ***complanar***, *i.e.* parallel to the same plane, if there exists a direction $(l, m, n)$ perpendicular to all three. This will be the case if the equations
$$l_1 l + m_1 m + n_1 n = 0,$$
$$l_2 l + m_2 m + n_2 n = 0,$$
$$l_3 l + m_3 m + n_3 n = 0$$
have solutions not all zero; hence the *condition of complanarity*
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

## EXERCISES

1. Find the length and direction cosines of the vector drawn from the point $(5, -2, 1)$ to the point $(4, 8, -6)$; from the point $(a, b, c)$ to the point $(-a, -b, -c)$; from $(-a, -b, -c)$ to $(a, b, c)$.

2. Show that when two lines with direction cosines $l$, $m$, $n$ and $l'$, $m'$, $n'$, respectively, are parallel, $ll' + mm' + nn' = \pm 1$.

3. Show that when two lines with direction cosines proportional to $a$, $b$, $c$ and $a'$, $b'$, $c'$ are perpendicular $aa' + bb' + cc' = 0$; and when the lines are parallel $a/a' = b/b' = c/c'$.

4. Show that the points $(5, 2, -3)$, $(6, 1, 4)$, $(-2, -3, 6)$, $(-1, -4, 13)$ are the vertices of a parallelogram.

5. Show by direction cosines that the points $(6, -3, 5)$, $(8, 2, 2)$, $(4, -8, 8)$ lie in a line.

6. Find the angle between the vectors from $(5, 8, -2)$ to $(-2, 6, -1)$ and from $(8, 3, 5)$ to $(1, 1, -6)$.

7. Find the angles of the triangle whose vertices are $(5, 2, 1)$, $(0, 3, -1)$, $(2, -1, 7)$.

8. Find the direction cosines of a line which is perpendicular to two lines whose direction cosines are proportional to $2, -3, 4$, and $5, 2, -1$, respectively.

9. Derive the formula of § 204 by taking on each line a vector of unit length, $OP_1$ and $OP_2$, and expressing the distance $P_1P_2$ first by the cosine law of trigonometry, then by § 196, and equating these expressions.

10. Find the rectangular components of a force of 12 lb. acting along a line inclined at 60° to $Ox$ and at 45° to $Oy$.

11. Find the resultant of the forces $OP_1$, $OP_2$, $OP_3$, $OP_4$ if the coordinates of $P_1$, $P_2$, $P_3$, $P_4$, with $O$ as origin, are $(3, -1, 2)$, $(2, 2, -1)$, $(-1, 2, 1)$, $(-2, 3, -4)$.

12. If any number of vectors, applied at the origin, are given by the coordinates $x$, $y$, $z$ of their extremities, the length of the resultant $R$ is $\sqrt{(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2}$ (see Ex. 9, p. 20), and its direction cosines are $\Sigma x/R$, $\Sigma y/R$, $\Sigma z/R$.

13. A particle at one vertex of a cube is acted upon by seven forces represented by the vectors from the particle to the other seven vertices; find the magnitude (length) and direction of the resultant.

14. If four forces acting on a particle are parallel and proportional to the sides of a quadrilateral, the forces are in equilibrium, *i.e.* their resultant is zero. Similarly for any closed polygon.

**208. Translation of Coordinate Trihedral.** Let $x$, $y$, $z$ be the coordinates of any point $P$ with respect to the trihedral formed by the axes $Ox$, $Oy$, $Oz$ (Fig. 113). If parallel axes $O_1x_1$, $O_1y_1$, $O_1z_1$ be drawn through any point $O_1(a, b, c)$, and if $x_1$, $y_1$, $z_1$ are the coordinates of $P$ with respect to the new trihedral $O_1x_1y_1z_1$, then the relations between the old coordinates $x, y, z$, and the new coordinates $x_1, y_1, z_1$ of one and the same point $P$ are evidently

$$x = a + x_1, \quad y = b + y_1, \quad z = c + z_1.$$

The coordinate trihedral has thus been given a *translation*, represented by the vector $OO_1$. This operation is also called a *transformation to parallel axes* through $O_1$.

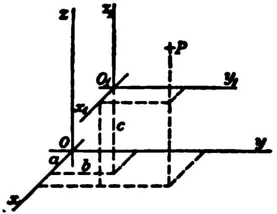

FIG. 113

**209. Area of a Triangle.** Any two vectors $OP_1$, $OP_2$ drawn from the origin determine a triangle $OP_1P_2$, whose area $A$ can easily be expressed if the lengths $r_1$, $r_2$ and direction cosines of the vectors are given. For, denoting the angle $P_1OP_2$ by $\psi$, we have for the area $A$:

$$A = \tfrac{1}{2}\, r_1r_2 \sin\psi,$$

where $\sin\psi$ can be expressed in terms of the direction cosines by § 206.

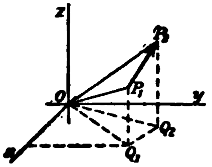

FIG. 114

**210. Moment of a Force.** Such areas are used in mechanics to represent the *moments* of forces. The moment of a force about a point $O$ is defined as the product of the force into the perpendicular distance of $O$ from the line of action of the force. Thus, if the vector $P_1P_2$ (Fig. 115) represent a force (in magnitude, direction, and sense) the moment of this force about the origin $O$ is equal to twice the area of the triangle $OP_1P_2$, *i.e.* to the area of the parallelogram $OP_1P_2P_3$, where $OP_3$ is a vector equal to the vector $P_1P_2$.

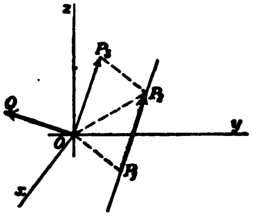

FIG. 115

It is often more convenient to represent this moment not by such an area, but by a *vector* $OQ$, drawn from $O$ at right angles to the triangle, and of a length equal to the number that represents the moment. If the body on which the force acts could turn freely about this perpendicular, the moment would represent the turning effect of the force $P_1P_2$.

The *sense* of this vector that represents the moment is taken so as to make the vector point toward that side of the plane of the triangle from which the force $P_1P_2$ is seen to turn counterclockwise.

**211.** If we square the expression found in § 209 for the area of the triangle $OP_1P_2$ and substitute for $\sin^2\psi$ its value from § 206, we find:

$$A^2 = \tfrac{1}{4} r_1^2 r_2^2 \left( \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 \right).$$

Hence $A^2$ is the sum of the squares of the three quantities

$$A_x = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, \quad A_y = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, \quad A_z = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix},$$

which have a simple geometrical and mechanical interpretation. For, as the coordinates of $P_1$, $P_2$ are

$$x_1 = l_1 r_1, \quad y_1 = m_1 r_1, \quad z_1 = n_1 r_1,$$
$$x_2 = l_2 r_2, \quad y_2 = m_2 r_2, \quad z_2 = n_2 r_2,$$

we have, *e.g.*,

$$A_z = \tfrac{1}{2} \begin{vmatrix} l_1 r_1 & m_1 r_1 \\ l_2 r_2 & m_2 r_2 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix};$$

and as $x_1$, $y_1$ and $x_2$, $y_2$ are the coordinates of the projections $Q_1$, $Q_2$ of $P_1$, $P_2$ on the plane $Oxy$, $A_z$ represents (§ 12) the area of the triangle $OQ_1Q_2$, *i.e. the projection on the plane Oxy of the area* $OP_1P_2$. Similarly, $A_x$ and $A_y$ are the projections of the area $OP_1P_2$ on the planes $Oyz$ and $Ozx$, respectively. As any three mutually rectangular planes can be taken as coordinate trihedrals, our formula $A^2 = A_x^2 + A_y^2 + A_z^2$ means that *the square of the area of any triangle is equal to the sum of the squares of its projections on any three mutually rectangular planes.*

In mechanics, $2A_z$ is the moment of the projection $Q_1Q_2$ of the force $P_1P_2$ about $O$, or what is by definition the same thing, the *moment of* $P_1P_2$ *about the axis Oz*. Similarly, for $2A_x$, $2A_y$. The proposition means, therefore, that the moments of $P_1P_2$ about the axes $Ox$, $Oy$, $Oz$ laid off as vectors along these axes can be regarded as the rectangular components of the moment of $P_1P_2$ about the point $O$; in other words, $2A_x$, $2A_y$, $2A_z$ are the components along $Ox$, $Oy$, $Oz$ of that vector $2A$ (§ 210) which represents the moment of $P_1P_2$ about $O$.

# CHAPTER XI

# THE PLANE AND THE STRAIGHT LINE

## PART I. THE PLANE

**214. Locus of One Equation.** In plane analytic geometry any equation between the coordinates $x$, $y$ or $r$, $\phi$ of a point in general represents a plane curve. In particular, an equation of the first degree in $x$ and $y$ represents a straight line (§ 30); an equation of the second degree in $x$ and $y$ in general represents a conic section (§ 154).

In solid analytic geometry any equation between the coordinates $x$, $y$, $z$ or $r$, $\theta$, $\phi$ of a point in general represents a ***surface.*** Thus, if any equation in $x$, $y$, $z$,

$$F(x, y, z) = 0,$$

be imagined solved for $z$ so as to take the form

$$z = f(x, y),$$

we can find from this equation to every point $(x, y)$ in the plane $Oxy$ one or more ordinates $z$ (which may of course be real or imaginary), and the ***locus*** formed by the extremities of the real ordinates will in general form a surface. It may however happen *in particular cases* that the locus of the equation $F(x, y, z) = 0$, *i.e.* the totality of all those points whose coordinates $x$, $y$, $z$ when substituted in the equation satisfy it, consists only of isolated points, or forms a curve, or that there are no real points satisfying the equation.

Similar considerations apply to an equation in polar coordinates

$$F(r, \theta, \phi) = 0.$$

**215. Locus of Two Simultaneous Equations.** Two simultaneous equations in $x$, $y$, $z$ (or in the polar coordinates $r$, $\theta$, $\phi$) will in general represent a *curve* in space, namely, the intersection of the two surfaces represented by the two equations separately.

Thus, in the present chapter, we shall see that an equation of the first degree in $x$, $y$, $z$ represents a plane and that therefore two such equations represent a straight line, the intersection or the two planes. In chapters XII and XIII we shall discuss loci represented by equations of the second degree, which are called *quadric surfaces.*

**216. Equation of a Plane.** *Every equation of the first degree in $x$, $y$, $z$ represents a plane.* The plane is defined as a surface such that the line joining any two of its points lies completely in the surface. We have therefore to show that if the general equation of the first degree

$$Ax + By + Cz + D = 0 \tag{1}$$

is satisfied by the coordinates of any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, *i.e.* if

$$\begin{cases} Ax_1 + By_1 + Cz_1 + D = 0, \\ Ax_2 + By_2 + Cz_2 + D = 0, \end{cases} \tag{2}$$

then (1) is satisfied by the coordinates of *every* point $P(x, y, z)$ of the line $P_1P_2$.

Now, by § 200, the coordinates of every point of the line $P_1P_2$ can be expressed in the form

$$x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ z = z_1 + k(z_2 - z_1),$$

where $k$ is the ratio in which $P$ divides $P_1P^2$, *i.e.*

$$k = P_1P/P_1P_2.$$

We have therefore to show that

$$A[x_1 + k(x_2 - x_1)] + B[y_1 + k(y_2 - y_1)] + C[z_1 + k(z_2 - z_1)] + D = 0,$$

whatever the value of $k$. Adding and subtracting $kD$, we can write this equation in the form

$$(1-k)(Ax_1+By_1+Cz_1+D)+k(Ax_2+By_2+Cz_2+D)=0;$$

and this is evidently true for any $k$, owing to the conditions (2).

**217. Essential Constants.** The equation (1) will still represent the same plane when multiplied by any constant different from zero. Since $A$, $B$, $C$ cannot all three be zero, we can divide (1) by one of these constants; it will then contain not more than three arbitrary constants. We say therefore that the general equation of a plane contains *three essential constants.* This corresponds to the geometrical fact that a plane can, in a variety of ways, be determined by three conditions, such as the conditions of passing through three points.

**218. Special Cases.** If, in equation (1), $D=0$, the plane evidently passes through the origin.

If, in equation (1), $C=0$, so that the equation is of the form $Ax+By+D=0$, this equation represents the plane perpendicular to the plane $Oxy$ and passing through the line whose equation in the plane $Oxy$ is $Ax+By+D=0$. For, the equation $Ax+By+D=0$ is satisfied by the coordinates of all points $(x, y, z)$ whose $x$ and $y$ are connected by the relation $Ax+By+D=0$ and whose $z$ is arbitrary, but it is not satisfied by the coordinates of any other points. Similarly, if $B=0$ in (1), the plane is perpendicular to $Ozx$; if $A=0$, the plane is perpendicular to $Oyz$.

If $B=0$ and $C=0$ in (1), the equation obviously represents a plane perpendicular to the axis $Ox$; and similarly when $C$ and $A$, or $A$ and $B$ are zero.

Notice that the line of intersection of (1) with the plane $Oxy$, for instance, is represented by the simultaneous equations

$$Ax+By+Cz+D=0, \quad z=0.$$

**219. Intercept Form.** If $D \neq 0$, the equation (1) can be divided by $D$; it then assumes the form

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0.$$

If $A$, $B$, $C$ are all different from zero, this equation can be written

$$\frac{x}{-D/A} + \frac{y}{-D/B} + \frac{z}{-D/C} = 1,$$

or, putting $-D/A = a$, $-D/B = b$, $-D/C = c$:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3}$$

In this equation, called the ***intercept form*** of the equation of a plane, the constants $a$, $b$, $c$ are the intercepts made by the plane on the axes $Ox$, $Oy$, $Oz$ respectively. For, putting, for instance, $y = 0$ and $z = 0$, we find $x = a$; etc.

**220. Plane through Three Points.** If the plane

$$Ax + By + Cz + D = 0$$

is to pass through the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, the three conditions

$$\begin{aligned} Ax_1 + By_1 + Cz_1 + D &= 0, \\ Ax_2 + By_2 + Cz_2 + D &= 0, \\ Ax_3 + By_3 + Cz_3 + D &= 0 \end{aligned}$$

must be satisfied. Eliminating $A$, $B$, $C$, $D$ between the *four* preceding equations, as in § 55, we find the equation of the plane passing through the three points in the form

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

### EXERCISES

**1.** Find the intercepts made by the following planes:

(a) $4x + 12y + 3z = 12$; (b) $15x - 6y + 10z + 30 = 0$;
(c) $x - y + z - 1 = 0$; (d) $x + 2y + 3z + 4 = 0$.

**2.** Interpret the following equations:

(a) $x + y + z = 1$; (b) $5y - 3z = 12$,
(c) $x + y = 0$; (d) $5y + 12 = 0$.

**3.** Find the plane determined by the points $(2, 1, 3)$, $(1, -5, 0)$, $(4, 6, -1)$.

**4.** Write down the equation of the plane whose intercepts are $3, 2, -5$.

**5.** Find the intercepts of the plane passing through the points $(3, -1, 4)$, $(6, 2, -3)$, $(-1, -2, -3)$.

**6.** If planes are parallel to and a distance $a$ from the coordinate planes, what are their intercepts? What are their equations?

**7.** Show that the four points $(4, 3, 3)$, $(4, -3, -9)$, $(0, 0, 3)$, $(2, 1, 2)$ lie in a plane and find its equation.

**221. Normal Form.** The position of a plane in space is fully determined by the length $p = ON$ (Fig. 118) of the perpendicular let fall from the origin on the plane and the direction cosines $l, m, n$ of this perpendicular regarded as a vector $ON$. Let $P$ be any point of the plane and $OQ = x$, $QR = y$, $RP = z$ its coordinates; as the projection of the open polygon $OQRP$ on $ON$ is equal to $ON$ (§ 199) we have

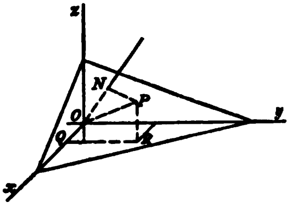

FIG. 118

$$lx + my + nz = p. \tag{4}$$

This equation is called the *normal form* of the equation of a plane. Observe that the number $p$ is always positive, being the distance of the plane from the origin, or the length of the vector $ON$. Hence $lx + my + nz$ is always positive.

**222. Reduction to the Normal Form.** The equation $Ax + By + Cz + D = 0$ is in general not of the form $ly+my+nz=p$ since in the latter equation the coefficients of $x$, $y$, $z$, being the direction cosines of a vector, have the property that the sum of their squares is equal to 1, while $A^2 + B^2 + C^2$ is in general not equal to 1. But the general equation can be reduced to the normal form by multiplying it by a constant factor $k$ properly chosen. The equation

$$kAx + kBy + kCz + kD = 0$$

evidently represents the same plane as does the equation $Ax + By + Cz + D = 0$; and we can select $k$ so that

$$(kA)^2 + (kB)^2 + (kC)^2 = 1, \quad \text{viz. } k = \frac{1}{\pm\sqrt{A^2+B^2+C^2}}.$$

As in the normal form the right-hand member $p$ is positive (§ 221) the sign of the square root should be selected so that $kD$ becomes negative.

*The normal form is therefore obtained by dividing the equation* $Ax + By + Cz + D = 0$ *by* $\pm\sqrt{A^2 + B^2 + C^2}$ *according as* $D$ *is negative or positive.*

It follows at the same time that the direction cosines of any normal to the plane $Ax + By + Cz + D = 0$ are proportional to $A$, $B$, $C$, viz.

$$l = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad m = \frac{B}{\pm\sqrt{A^2+B^2+C^2}},$$

$$n = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}},$$

and that the distance of the plane from the origin is

$$p = \frac{-D}{\pm\sqrt{A^2+B^2+C^2}},$$

the upper sign of the square root to be used when $D$ is negative, the lower when $D$ is positive.

**223. Distance of Point from Plane.** Let $lx + my + nz = p$ be the equation of a plane in the normal form, $P_1(x_1, y_1, z_1)$ any point not on this plane (Fig. 119). The projection $OS$ of the vector $OP_1$ on the normal to the plane being equal to the sum of the projections of its components $OQ = x_1$, $QR = y_1$, $RP_1 = z_1$, we have

$$OS = lx_1 + my_1 + nz_1.$$

Hence the distance $d$ of $P_1$ from the plane, which is equal to $NS$, will be

$$d = OS - ON = lx_1 + my_1 + nz_1 - p.$$

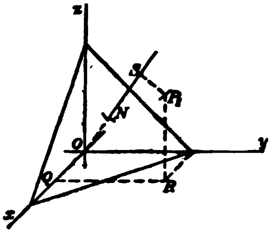

FIG. 119

If this expression is negative, the point $P_1$ lies on the same side of the plane as does the origin; if it is positive, the point $P_1$ lies on the opposite side of the plane. Any plane thus divides space into two regions, in one of which the distance of every point from the plane is positive, while in the other the distance is negative. If the plane does not pass through the origin, the region containing the origin is the negative region; if it does, either side can be taken as the positive side.

To find the distance of a point $P_1(x_1, y_1, z_1)$ from a plane given in the general form

$$Ax + By + Cz + D = 0,$$

we have only to reduce the equation to the normal form (§ 223) and then to substitute for $x$, $y$, $z$ the coordinates $x_1$, $y_1$, $z_1$ of $P_1$; thus

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

the square root being taken with + or − according as $D$ is negative or positive.

Notice that $d$ is the distance from the plane to the point $P_1$, not from $P_1$ to the plane.

**224. Angle between Two Planes.** As two intersecting planes make two angles whose sum $= \pi$, we shall, to avoid any ambiguity, define the angle between the planes as the angle between the perpendiculars (regarded as vectors) drawn from the origin to the two planes.

If the equations of the planes are given in the normal form,

$$l_1x + m_1y + n_1z = p_1,$$
$$l_2x + m_2y + n_2z = p_2,$$

we have, by § 204, for the angle $\psi$ between the planes:

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

If the equations of the planes are in the general form,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

we find by reducing to the normal form (§ 222):

$$\cos\psi = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}\cdot\pm\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

**225. Bisecting Planes.** To find the equations of the two planes that bisect the angles formed by two intersecting planes given in the normal form,

$$l_1x + m_1y + n_1z - p_1 = 0,\quad l_2x + m_2y + n_2z - p_2 = 0,$$

observe that for any point in either bisecting plane its distances from the two given planes must be equal in absolute value. Hence the equations of the required planes are

$$l_1x + m_1y + n_1z - p_1 = \pm(l_2x + m_2y + n_2z - p_2).$$

To distinguish the two planes, observe that for the plane that bisects that pair of vertical angles which contains the origin the perpendicular distances are in the one angle both positive, in the other both negative; hence the plus sign gives this bisecting plane.

If the equations of the planes are given in the general form,

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0,$$

first reduce the equations to the normal form (§ 222).

### EXERCISES

1. A line is drawn from the origin perpendicular to the plane $x - y - 5z - 10 = 0$; what are the direction cosines of this line?

2. Find the distance from the origin to the plane $2x + 2y - z = 6$.

3. Find the distances of the following planes from the origin:

(a) $3x - 4y + 5z - 8 = 0$, (b) $x + y + z = 0$,

(c) $2y - 5z = 3$, (d) $3x - 4y + 5 = 0$.

4. Find the distances from the following planes to the point $(2, 1, -3)$:

(a) $3x + 5y - 6z = 8$, (b) $2x - 3y - z = 0$, (c) $x + y + z = 0$.

5. Find the plane through the point (4, 8, 1) which is perpendicular to the radius vector of this point; also the parallel plane whose distance from the origin is 10 and in the same sense.

6. Find the plane through the point $(-1, 2, -4)$ that is parallel to the plane $4x - 3y + 2z = 8$; what is the distance between these planes?

7. Find the distance between the planes $4x - 5y - 2z = 6$, $4x - 5y - 2z + 8 = 0$.

8. Are the points $(6, 1, -4)$ and $(4, -2, 3)$ on the same side of the plane $2x + 3y - 5z + 1 = 0$?

9. Write down the equation of the plane equally inclined to the axes and at the distance $p$ from the origin.

10. Show that the relation between the distance $p$ from the origin to a plane and the intercepts $a$, $b$, $c$ is $1/a^2 + 1/b^2 + 1/c^2 = 1/p^2$.

11. Show that the locus of the points equally distant from the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is a plane that bisects $P_1P_2$ at right angles.

12. Find the equations of the planes bisecting the angles: (a) between the planes $x + y + z - 3 = 0$, $2x - 3y + 4z + 3 = 0$; (b) between the planes $2x - 2y - z = 8$, $x + 2y - 2z = 6$.

**226. Volume of a Tetrahedron.** The volume of the tetrahedron whose vertices are the points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ can be expressed in terms of the coordinates of the points. The equation of the plane determined by the points $P_2$, $P_3$, $P_4$ is (§ 220)

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Now the altitude $d$ of the tetrahedron is the distance from this plane to the point $P_1(x_1, y_1, z_1)$, *i.e.* (§ 223)

$$d = \frac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2}}.$$

But the denominator is seen immediately to represent twice the area of the triangle with vertices $P_2$, $P_3$, $P_4$ (Ex. 9, p. 203), *i.e.* twice the base of the tetrahedron. Denoting the base by $B$, we then have

$$2\,Bd = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The volume of the tetrahedron is $V = \frac{1}{3}Bd$, and therefore

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

**227. Simultaneous Linear Equations.** Two simultaneous equations of the first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

represent in general the line of intersection of the two planes represented by the two equations separately. For, the coordinates of every point of this line, and those of no other point, satisfy both equations. See § 215 and §§ 231–232.

*Three simultaneous equations* of the first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$
$$A_3x + B_3y + C_3z + D_3 = 0,$$

determine in general the point of intersection of the three planes. The coordinates of this point are found by solving the three equations for $x, y, z$. But it may happen that the three planes have no common point, as when the three lines of intersection are parallel, or when the three planes are parallel; and it may happen that the planes have an infinite number of points in common, as when two of the planes, or all three, coincide, or when the three planes pass through one and the same line.

Four planes will in general have no point in common. If they do, *i.e.* if there exists a point $(x_1, y_1, z_1)$ satisfying the four equations

$$A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0,$$
$$A_2x_1 + B_2y_1 + C_2z_1 + D_2 = 0,$$
$$A_3x_1 + B_3y_1 + C_3z_1 + D_3 = 0,$$
$$A_4x_1 + B_4y_1 + C_4z_1 + D_4 = 0,$$

we can eliminate $x_1$, $y_1$, $z_1$, 1 between these equations so that we find the condition

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0.$$

### EXERCISES

1. Find the volume of the tetrahedron whose vertices are (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c).

2. Find the volumes of the tetrahedra whose vertices are the following points:

   (a) $(7, 0, 6)$, $(3, 2, 1)$, $(-1, 0, 4)$, $(3, 0, -2)$.
   (b) $(3, 0, 1)$, $(0, -8, 2)$, $(4, 2, 0)$, $(0, 0, 10)$.
   (c) $(2, 1, -3)$, $(4, -2, 1)$, $(3, -7, -4)$, $(5, 1, 8)$.

3. Find the coordinates of the points in which the following planes intersect:

   (a) $2x + 5y + z - 2 = 0$, $x + 5y + z = 0$, $3x - 3y + 2z - 12 = 0$,
   (b) $2x + y + z = a + b + c$, $4x - 2y + z = 2a - 2b + c$, $6x - y = 3a - b$.

4. Show that the four planes $5x - 3y - z = 0$, $4x - 2y + z = 3$, $3x + 2y - 6z = 6$, $x + y + z = 6$ pass through the same point. What are the coordinates of this point?

5. Show that the four planes $4x + y + z + 4 = 0$, $x + 2y - z + 3 = 0$, $y - 5z + 14 = 0$, $x + y + z - 2 = 0$ have a common point.

6. Show that the locus of a point the sum of whose distances from any number of fixed planes is constant is a plane.

**228. Pencil of Planes.** All the planes that pass through one and the same line are said to form a *pencil* of planes, and their common line is called the ***axis*** of the pencil.

If the equations of any two non-parallel planes are given, say

$$(1)\qquad \begin{aligned} A_1x + B_1y + C_1z + D_1 = 0,\\ A_2x + B_2y + C_2z + D_2 = 0, \end{aligned}$$

then the equation of any other plane of the pencil having their intersection as axis can be written in the form

$$(2)\qquad (A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0,$$

where $k$ is a constant whose value determines the position of the plane in the pencil.

For, this equation (2) being of the first degree in $x$, $y$, $z$ certainly represents a plane; and the coordinates of the points of the line of intersection of the two given planes (1), since they satisfy each of the equations (1), must satisfy the equation (2) so that the plane (2) passes through the axis of the pencil.

**229. Sheaf of Planes.** All the planes that pass through one and the same point are said to form a *sheaf* of planes, and their common point is called the ***center*** of the sheaf.

If the equations of any three planes, not of the same pencil, are given, say

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$
$$A_3x + B_3y + C_3z + D_3 = 0,$$

then the equation of any other plane of the sheaf having their point of intersection as center can be written in the form

$$(A_1x + B_1y + C_1z + D_1) + k_1(A_2x + B_2y + C_2z + D_2)$$
$$+ k_2(A_3x + B_3y + C_3z + D_3) = 0,$$

where $k_1$ and $k_2$ are constants whose values determine the position of the plane in the sheaf.

The proof is similar to that of § 228.

**230. Non-linear Equations Representing Several Planes.** When two planes are given, say

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

then the equation

$$(A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2) = 0,$$

obtained by equating to zero the product of the left-hand members (the right-hand members being reduced to zero), is satisfied by all the points of the first given plane as well as all the points of the second given plane, and by no other points.

The product equation is therefore said to represent the two given planes. The equation is of the second degree.

Similarly, by equating to zero the product of the left-hand members of the equations of three or more planes (the right-hand members being zero) we obtain a single equation representing all these planes. An equation of the $n$th degree *may*, therefore, represent $n$ planes; it will do so if its left-hand member can be resolved into $n$ linear factors with real coefficients.

### EXERCISES

1. Find the plane that passes through the line of intersection of the planes $5x - 3y + 4z - 35 = 0$, $x + y - z = 0$ and through $(4, -3, 2)$.

2. Show that the planes $3x - 2y + 5z + 2 = 0$, $x + y - z - 5 = 0$, $6x + y + 2z - 13 = 0$ belong to the same pencil.

3. Show that the following planes belong to the same sheaf and find the coordinates of the center of the sheaf: $6x + y - 4z = 0$, $x + y + z = 5$, $2x - 4y - z = 10$, $2x + 3y + z = 4$.

4. What planes are represented by the following equations?
(*a*) $x^2 - 6x + 8 = 0$, (*b*) $y^2 - 9 = 0$, (*c*) $x^2 - z^2 = 0$, (*d*) $x^2 - 4xy = 0$.

5. Find the cosine of the angle between the following pairs of planes:
(*a*) $4x - 3y - z = 6$, $x + y - z = 8$; (*b*) $2x + 7y + 4z = 2$, $x - 9y - 2z = 12$.

6. Show that the following pairs of planes are either parallel or perpendicular:
(*a*) $3x - 2y + 5z = 0$, $2x + 3y = 8$; (*b*) $5x + 2y - z = 6$, $10x + 4y - 2z = 3$; (*c*) $x + y - 2z = 3$, $x + y + z = 11$; (*d*) $x - 2y - z = 8$, $3x - 6y - 3z = 5$.

7. Find the plane that is perpendicular to the segment joining the points $(3, -4, 6)$ and $(2, 1, -3)$ at its midpoint.

8. Show that the planes $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$ are parallel (on the same or opposite sides of the origin) if

$$\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} = \pm 1.$$

9. A cube whose edges have the length $a$ is referred to a coordinate trihedral, the origin being taken at the center of a face and the axes parallel to the edges of the cube. Find the equations of the faces.

**10.** Show that the plane through the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ and perpendicular to the plane $Ax + By + Cz + D = 0$ can be represented by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ A & B & C & 0 \end{vmatrix} = 0.$$

**11.** Find those planes of the pencil $4x - 3y + 5z = 8$, $2x + 3y - z = 4$ which are perpendicular to the coordinate planes.

**12.** Find the plane that is perpendicular to the plane $2x + 3y - z = 1$ and passes through the points $(1, 1, -1)$, $(3, 4, 2)$.

**13.** Find the plane that is perpendicular to the planes $4x - 3y + z = 6$, $2x + 3y - 5z = 4$ and passes through the point $(4, -1, 5)$.

**14.** Show that the conditions that three planes $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$, $A_3x + B_3y + C_3z + D_3 = 0$ belong to the same pencil, are

$$\frac{A_1 + kA_2}{A_3} = \frac{B_1 + kB_2}{B_3} = \frac{C_1 + kC_2}{C_3} = \frac{D_1 + kD_2}{D_3};$$

or, putting these fractions equal to $s$ and eliminating $k$ and $s$,

$$\begin{vmatrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix} = \begin{vmatrix} C_1 & D_1 & A_1 \\ C_2 & D_2 & A_2 \\ C_3 & D_3 & A_3 \end{vmatrix} = \begin{vmatrix} D_1 & A_1 & B_1 \\ D_2 & A_2 & B_2 \\ D_3 & A_3 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

(Verify Ex. 2 by using these conditions.)

**15.** Find the equations of the faces of a right pyramid, with square base of side $2a$ and with altitude $h$, the origin being taken at the center of the base, the axis $Oz$ through the opposite vertex, and the axes $Ox$, $Oy$ parallel to the sides of the base.

**16.** Homogeneous substances passing from a liquid to a solid state tend to form crystals; *e.g.* an ideal specimen of ammonium alum has the form of a regular octahedron. Find the equations of the faces of such a crystal of edge $a$ if the origin is taken at the center and the axes through the vertices, and determine the angle between two faces.

**17.** Find the angles between the lateral faces of a right pyramid whose base is a regular hexagon of side $a$ and whose altitude is $h$.

## PART II. THE STRAIGHT LINE

**231. Determination of Direction Cosines.** Two simultaneous linear equations (§ 227),

(1) $$Ax+By+Cz+D=0,\quad A'x+B'y+C'z+D'=0,$$

represent a line, namely, the intersection of the two planes represented by the two equations separately, provided the two planes are not parallel.

To obtain the direction cosines $l$, $m$, $n$ of this line observe that the line, since it lies in each of the two planes, is perpendicular to the normal of each plane. Now, by § 222 the direction cosines of these normals are proportional to $A$, $B$, $C$ and $A'$, $B'$, $C'$, respectively. We have therefore

$$Al+Bm+Cn=0,\quad A'l+B'm+C'n=0,$$

whence

$$l:m:n=\begin{vmatrix} B & C \\ B' & C' \end{vmatrix} : \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} : \begin{vmatrix} A & B \\ A' & B' \end{vmatrix}.$$

The direction cosines themselves are then found by dividing each of these determinants by the square root of the sum of their squares.

**232. Intersecting Lines.** The two lines

$$\left.\begin{matrix} A_1x+B_1y+C_1z+D_1=0, \\ A_1'x+B_1'y+C_1'z+D_1'=0 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} A_2x+B_2y+C_2z+D_2=0, \\ A_2'x+B_2'y+C_2'z+D_2'=0 \end{matrix}\right.$$

will intersect if, and only if, the four planes represented by these equations have a common point. By § 227, the condition for this is

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_1' & B_1' & C_1' & D_1' \\ A_2 & B_2 & C_2 & D_2 \\ A_2' & B_2' & C_2' & D_2' \end{vmatrix}=0.$$

**233. Special Forms of Equations.** For many purposes it is convenient to represent a line by means of one of its points and its direction cosines, or by means of two of its points. Let the line be called $\lambda$.

If $(x_1, y_1, z_1)$ is a given point of $\lambda$ and $l, m, n$ are the direction cosines of $\lambda$, then every point $(x, y, z)$ of $\lambda$ must satisfy the relations (§ 203):

$$(2) \qquad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z^1}{n}.$$

In these equations, $l, m, n$, *can evidently be replaced by any three numbers proportional to* $l, m, n$. Thus, if $(x_2, y_2, z_2)$ be any point of $\lambda$ different from $(x_1, y_1, z_1)$, we have the continued proportion

$$x_2 - x_1 : y_2 - y_1 : z_2 - z_1 = l : m : n;$$

hence *the equations of the line through the two points* $(x_1, y_1, z_1)$ *and* $(x_2, y_2, z_2)$ are:

$$(3) \qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

If, for the sake of brevity, we put $x_2 - x_1 = a$, $y_2 - y_1 = b$, $z_2 - z_1 = c$, we can write the equations of the line in the form

$$(4) \qquad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where $a, b, c$, are proportional to $l, m, n$, and can be regarded as the components of a vector parallel to the line.

The equations (3) also follow directly by eliminating $k$ between the equations of § 200, namely,

$$(5) \quad x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ z = z_1 + k(z_2 - z_1).$$

These equations which, with a variable $k$, represent any point of the line through $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are called the ***parameter equations*** of the line.

**234. Projecting Planes of a Line.** Each of the forms (2), (3), (4), which are not essentially different, furnishes three linear equations; thus (4) gives:

$$\frac{y-y_1}{b}=\frac{z-z_1}{c},\quad \frac{z-z_1}{c}=\frac{x-x_1}{a},\quad \frac{x-x_1}{a}=\frac{y-y_1}{b};$$

but these three equations are equivalent to only two, since from any two the third follows immediately.

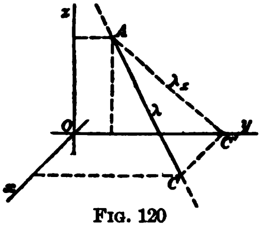

FIG. 120

The first of these equations, which can be written in the form

$$cy-bz-(cy_1-bz_1)=0,$$

represents, since it does not contain $x$ (§ 218), a plane perpendicular to the plane $Oyz$; and as this plane must contain the line $\lambda$ it is *the plane $CC'A$ that projects $\lambda$ on the plane $Oyz$* (Fig. 120). Similarly the other two equations represent the planes that project $\lambda$ on the coordinate planes $Ozx$ and $Oxy$. Any two of these equations represent the line $\lambda$ as the intersection of two of these projecting planes.

At the same time the equation

$$\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

can be interpreted as representing a line in the plane $Oyz$, viz. the intersection of the projecting plane with the plane $x=0$. This line ($AC'$ in Fig. 120) is *the projection $\lambda_x$ of $\lambda$ on the plane $Oyz$*. As the other two equations (4) can be interpreted similarly it appears that the equations (2), (3), or (4) represent the line $\lambda$ by means of its projections $\lambda_x$, $\lambda_y$, $\lambda_z$ on the three coordinate planes, just as is done in descriptive geometry. Any two of the projections are of course sufficient to determine the line.

**235. Determination of Projecting Planes.** To reduce the equations of a line $\lambda$ given in the form (1) to the form (4) we have only to eliminate between the equations (1) first one of the variables $x$, $y$, $z$, then another, so as to obtain two equations, each in only two variables (not the same in both).

The process will best be understood from an example. The line being given as the intersection of the planes

$$(a)\ 2x - 3y + z + 3 = 0,$$
$$(b)\ x + y + z - 2 = 0,$$

eliminate $z$ by subtracting ($b$) from ($a$) and eliminate $x$ by subtracting ($b$), multiplied by 2, from ($a$); this gives the line as the intersection of the planes

$$x - 4y + 5 = 0,$$
$$-5y - z + 7 = 0,$$

which are the projecting planes parallel to $Oz$ and $Ox$, *i.e.* the planes that project the line on $Oxy$ and $Oyz$. Solving for $y$ and equating the two values of $y$ we find:

$$\frac{x+5}{4} = \frac{y}{1} = \frac{z-7}{-5}.$$

The line passes therefore through the point $(-5, 0, 7)$ and has direction cosines proportional to $4, 1, -5$, viz.

$$l = \frac{4}{\sqrt{42}}, \quad m = \frac{1}{\sqrt{42}}, \quad n = -\frac{5}{\sqrt{42}}.$$

### EXERCISES

1. Write the equations of the line through the point $(-3, 1, 6)$ whose direction cosines are proportional to $3, 5, 7$.

2. Write the equations of the line through the point $(3, 2, -4)$ whose direction cosines are proportional to $5, -1, 3$.

3. Find the line through the point $(a, b, c)$ that is equally inclined to the axes of coordinates.

4. Find the lines that pass through the following pairs of points:

(*a*) (4, − 3, 1), (2, 3, 2), (*b*) (− 1, 2, 3), (8, 7, 1),

(*c*) (− 2, 3, − 4), (0, 2, 0), (*d*) (− 1, − 5, − 2), (− 3, 0, −1),

and determine the direction cosines of each of these lines.

5. Find the traces of the plane $2x - 3y - 4z = 6$ in the coordinate planes.

6. Write the equations of the line $2x - 3y + 5z - 6 = 0$, $x - y + 2z - 3 = 0$ in the form (4) and determine the direction cosines.

7. Put the line $4x - 3y - 6 = 0$, $x - y - z - 4 = 0$ in the form (4) and determine the direction cosines.

8. Find the line through the point (2, 1, − 3) that is parallel to the line $2x - 3y + 4z - 6 = 0$, $5x + y - 2z - 8 = 0$.

9. What are the projections of the line $5x - 3y - 7z - 10 = 0$, $x + y - 3z + 5 = 0$ on the coordinate planes?

10. Obtain the equations of the line through two given points by equating the values of $k$ obtained from § 200.

11. By § 222, the direction cosines of any line are proportional to the coefficients of $x$, $y$, and $z$ in the equation of a plane perpendicular to the line. Find a line through the point (3, 5, 8) that is perpendicular to the plane $2x + y + 3z = 5$.

**236. Angle between Two Lines.** The cosine of the angle $\psi$ between two lines whose direction cosines are $l_1, m_1, n_1$ and $l_2, m_2, n_2$ is, by § 204,

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

Hence if the lines are given in the form (4), say

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2},$$

we have

$$\cos\psi = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\pm\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\pm\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

If the lines are *parallel*, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2};$$

if they are *perpendicular*, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0;$$

and *vice versa*.

**237. Angle between Line and Plane.** Let the line and plane be given by the equations

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c},$$

$$Ax+By+Cz+D=0.$$

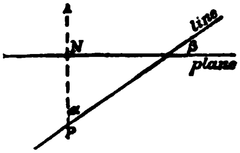

FIG. 121

The plane of Fig. 121 represents the plane through the given line perpendicular to the given plane. The angle $\beta$ between the given line and plane is the complement of the angle $\alpha$ between the line and any perpendicular $PN$ to the plane. Hence

$$\sin\beta=\frac{aA+bB+cC}{\pm\sqrt{a^2+b^2+c^2}\cdot\pm\sqrt{A^2+B^2+C^2}}.$$

The (necessary and sufficient) condition for *parallelism* of line and plane is

$$aA+bB+cC=0;$$

the condition of *perpendicularity* is

$$\frac{a}{A}=\frac{b}{B}=\frac{c}{C}.$$

**238. Line and Plane Perpendicular at Given Point.** If the plane

$$Ax+By+Cz+D=0$$

passes through the point $P_1(x_1,y_1,z_1)$, we must have

$$Ax_1+By_1+Cz_1+D=0.$$

Subtracting from the preceding equation, we have as *the equation of any plane through the point* $P_1(x_1,y_1,z_1)$:

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0.$$

The equations of any line through the same point are

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

If this line is perpendicular to the plane, we must have (§ 237): $a/A=b/B=c/C$. Hence the equations

$$\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C}$$

represent the line through $P_1(x_1,y_1,z_1)$ perpendicular to the plane $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$.

**239. Distance of a Point from a Line.** If the equations of the line $\lambda$ are given in the form

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

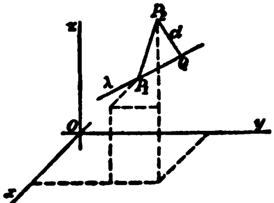

FIG. 122

where $(x_1, y_1, z_1)$ is a point $P_1$ of $\lambda$ (Fig. 122), the distance $d = QP_2$ of the point $P_2(x_2, y_2, z_2)$ from $\lambda$ can be found from the right-angled triangle $P_1QP_2$ which gives

$$d^2 = P_1P_2^2 - P_1Q^2,$$

by observing that

$$P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

while $P_1Q$ is the projection of $P_1P_2$ on $\lambda$. This projection is found (§ 199) as the sum of the projections of the components $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ of $P_1P_2$ on $\lambda$:

$$P_1Q = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Hence

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - [l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)]^2.$$

**240. Shortest Distance between Two Lines.** *Two lines* $\lambda_1$, $\lambda_2$ whose equations are given in the form

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

*will intersect* if their directions $(l_1, m_1, n_1)$, $(l_2, m_2, n_2)$, and the direction of the line joining the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ are complanar (§ 207), *i.e.* if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

If the lines $\lambda_1$, $\lambda_2$ do not intersect, their shortest distance $d$ is the distance of $P_2(x_2, y_2, z_2)$ from the plane through $\lambda_1$ parallel to $\lambda_2$. As this plane contains the directions of $\lambda_1$ and $\lambda_2$, the direction cosines of its normal are (§ 206) proportional to

$$\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix},\quad \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix},\quad \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix};$$

and as it passes through $P_1(x_1, y_1, z_1)$ its equation can be written in the form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Hence the ***shortest distance of the lines*** $\lambda_1, \lambda_2$ is:

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2}}.$$

As the denominator of this expression is equal to $\sin\psi$ (§ 206), we have

$$d \sin\psi = \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}.$$

## EXERCISES

**1.** Find the cosine of the angle between the lines

$$\frac{x-3}{2} = \frac{y-5}{3} = \frac{z+1}{4} \text{ and } \frac{x+1}{-1} = \frac{y-3}{2} = \frac{z+3}{3}.$$

**2.** Find the angle between the lines $3x - 2y + 4z - 1 = 0$, $2x + y - 3z + 10 = 0$, and $x + y + z = 6$, $2x + 3y - 5z = 8$.

**3.** Find the angle between the lines that pass through the points $(4, 2, 5)$, $(-2, 4, 3)$ and $(-1, 4, 2)$, $(4, -2, -6)$.

**4.** Find the angle between the line

$$\frac{x+1}{3} = \frac{y-2}{-5} = \frac{z+10}{3}$$

and a perpendicular to the plane $4x - 3y - 2z = 8$.

**5.** In what ratio does the plane $3x - 4y + 6z - 8 = 0$ divide the segment drawn from the origin to the point $(10, -8, 4)$.

**6.** Find the plane through the point $(2, -1, 3)$ perpendicular to the line

$$\frac{x-3}{4} = \frac{y+2}{3} = \frac{z-7}{-1}.$$

**7.** Find the plane that is perpendicular to the line $4x + y - z = 6$, $3x + 4y + 8z + 10 = 0$ and passes through the point $(4, -1, 3)$.

**8.** Find the plane through the origin perpendicular to the line

$$5x - 2y + z = 6,\quad 3x + y - 4z = 8.$$

**9.** Find the plane through the point $(4, -3, 1)$ perpendicular to the line joining the points $(3, 1, -6)$, $(-2, 4, 7)$.

**10.** Find the line through the point $(2, -1, 4)$ perpendicular to the plane $x - 2y + 4z = 6$.

**11.** Show that the lines $x/3 = y/-1 = z/-2$ and $x/4 = y/6 = z/3$ are perpendicular.

**12.** Show that the lines

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3} \text{ and } \frac{x-2}{-2} = \frac{y-3}{4} = \frac{z}{-6}$$

are parallel.

**13.** Find the angle between the line $3x - 2y - z = 4$, $4x + 3y - 3z = 6$ and the plane $x + y + z = 8$.

**14.** Find the lines bisecting the angles between the lines

$$\frac{x-a}{l_1} = \frac{y-b}{m_1} = \frac{z-c}{n_1} \text{ and } \frac{x-a}{l_2} = \frac{y-b}{m_2} = \frac{z-c}{n_2}.$$

**15.** Find the plane perpendicular to the plane $3x - 4y - z = 6$ and passing through the points $(1, 3, -2)$, $(2, 1, 4)$.

**16.** Find the plane through the point $(3, -1, 2)$ perpendicular to the line $2x - 3y - 4z = 7$, $x + y - 2z = 4$.

**17.** Find the plane through the point $(a, b, c)$ perpendicular to the line $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$.

**18.** Find the projection of the vector from $(3, 4, 5)$ to $(2, -1, 4)$ on the line that makes equal angles with the axes; and on the plane

$$2x - 3y + 4z = 6.$$

**19.** Find the distances from the following lines to the points indicated:

(a) $\frac{x}{3} = \frac{y-2}{5} = \frac{z+1}{2}$, $(0, 0, 0)$;

(b) $2x + y - z = 6$, $x - y + 4z = 8$, $(3, 1, 4)$;

(c) $2x + 3y + 5z = 1$, $3x - 6y + 3z = 0$, $(4, 1, -2)$.

**20.** Show that the equation of the plane determined by the line

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

and the point $P_2(x_2, y_2, z_2)$ can be written in the form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0.$$

**21.** Find the plane determined by the intersecting lines

$$\frac{x-3}{4}=\frac{y-5}{3}=\frac{z+1}{2} \text{ and } \frac{x-3}{1}=\frac{y-5}{2}=\frac{z+1}{3}.$$

**22.** Find the plane determined by the line

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c},$$

and its parallel through the point $P_2(x_2, y_2, z_2)$.

**23.** Given two non-intersecting lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1},\quad \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2};$$

find the plane passing through the first line and a parallel to the second; and the plane passing through the second line and a parallel to the first.

**24.** What is the condition that the two lines of Ex. 23 intersect?

**25.** Find the distance from the diagonal of a cube to a vertex not on the diagonal.

**26.** Find the distance between the lines given in Ex. 23.

**27.** Show that the locus of the points whose distances from two fixed planes are in constant ratio is a plane.

**28.** Show that the plane $(m-n)x+(n-l)y+(l-m)z=0$ contains the line $x/l=y/m=z/n$ and is perpendicular to the plane determined by the lines $x/m=y/n=z/l$ and $x/n=y/l=z/m$.

# CHAPTER XII

## THE SPHERE

**241. Spheres.** A sphere is defined as the locus of all those points that have the same distance from a fixed point.

Let $C(h, j, k)$ denote the center, and $r$ the radius, of a sphere; the necessary and sufficient condition that any point $P(x, y, z)$ has the distance $r$ from $C(h, j, k)$ is

$$(1) \qquad (x-h)^2+(y-j)^2+(z-k)^2=r^2.$$

This then is *the cartesian equation of the sphere of center $C(h, j, k)$ and radius $r$.*

If the center of the sphere lies in the plane $Oxy$, the equation becomes

$$(x-h)^2+(y-j)^2+z^2=r^2.$$

If the center lies on the axis $Ox$, the equation is

$$(x-h)^2+y^2+z^2=r^2.$$

The equation of a sphere about the origin as center is:

$$x^2+y^2+z^2=r^2.$$

**242. Expanded Form.** Expanding the squares in the equation (1), we find the equation of the sphere in the form

$$x^2+y^2+z^2-2hx-2jy-2kz+h^2+j^2+k^2-r^2=0.$$

This is an equation of the second degree in $x, y, z$; but it is of a particular form.

The ***general equation of the second degree*** in $x, y, z$ is

$$(2) \qquad Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy \\ +2Gx+2Hy+2Iz+J=0;$$

*i.e.* it contains a constant term $J$; three terms of the first degree, one in $x$, one in $y$, and one in $z$; and six terms of the second degree, one each in $x^2$, $y^2$, $z^2$, $yz$, $zx$, and $xy$.

If in (2) we have $D=E=F=0$, $A=B=C\neq 0$, it reduces, upon division by $A$, to the form

$$x^2+y^2+z^2+\frac{2\,G}{A}x+\frac{2\,H}{A}y+\frac{2\,I}{A}z+\frac{J}{A}=0,$$

which agrees with the above form of the equation of a sphere, apart from the notation for the coefficients.

**243. Determination of Center and Radius.** To determine the locus represented by the equation

$$(3) \qquad Ax^2+Ay^2+Az^2+2\,Gx+2\,Hy+2\,Iz+J=0,$$

where $A$, $G$, $H$, $I$, $J$ are any real numbers while $A\neq 0$, we divide by $A$ and complete the squares in $x$, $y$, $z$; this gives

$$\left(x+\frac{G}{A}\right)^2+\left(y+\frac{H}{A}\right)^2+\left(z+\frac{I}{A}\right)^2=\frac{G^2}{A^2}+\frac{H^2}{A^2}+\frac{I^2}{A^2}-\frac{J}{A}.$$

The left side represents the square of the distance of the point $(x, y, z)$ from the point $(-G/A, -H/A, -I/A)$; the right side is constant. Hence, if the right side is positive, the equation represents the sphere whose center has the coordinates $(-G/A, -H/A, -I/A)$, and whose radius is

$$r=\frac{1}{A}\sqrt{G^2+H^2+I^2-AJ}.$$

If, however, $G^2+H^2+I^2<AJ$, the equation is not satisfied by any point with real coordinates. If $G^2+H^2+I^2=AJ$, the equation is satisfied only by the coordinates of the point $(-G/A, -H/A, -I/A)$.

Thus *the equation of the second degree*

$$Ax^2+By^2+Cz^2+2\,Dyz+2\,Ezx+2\,Fxy$$
$$+2\,Gx+2\,Hy+2\,Iz+J=0,$$

*represents a sphere if, and only if,*

$$A=B=C\neq 0, \quad D=E=F=0, \quad G^2+H^2+I^2>AJ.$$

**244. Essential Constants.** The equation (1) of the sphere contains four constants: $h, j, k, r$. The equation (2) contains five constants of which, however, only four are essential since we can divide out by one of these constants. Thus dividing by $A$ and putting $2\,G/A = a$, $2\,H/A = b$, $2\,I/A = c$, $J/A = d$, the general equation (2) assumes the form

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$

with only the four essential constants $a$, $b$, $c$, $d$.

This fact corresponds to the possibility of determining a sphere geometrically, in a variety of ways, by four conditions.

### EXERCISES

**1.** Find the spheres with the following points as centers and with the indicated radii:

(*a*) $(4, -1, 2), 4$; (*b*) $(0, 0, 4), 4$; (*c*) $(2, -2, 1), 3$; (*d*) $(3, 4, 1), 7$.

**2.** Find the following spheres:

(*a*) with the points $(4, 2, 1)$ and $(3, -7, 4)$ as ends of a diameter;

(*b*) tangent to the coordinate planes and of radius $a$;

(*c*) with center at the point $(4, 1, 5)$ and passing through $(8, 3, -5)$.

**3.** Find the centers and the radii of the following spheres:

(*a*) $x^2 + y^2 + z^2 - 3\,x + 5\,y - 6\,z + 2 = 0$.

(*b*) $x^2 + y^2 + z^2 - 2\,bx + 2\,cz - b^2 - c^2 = 0$.

(*c*) $2\,x^2 + 2\,y^2 + 2\,z^2 + 3\,x - y + 5\,z - 11 = 0$.

(*d*) $x^2 + y^2 + z^2 - x - y - z = 0$.

**4.** Show that the equation $A(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0$, in which $J$ is variable, represents a family of concentric spheres.

**5.** Find the spheres that pass through the following points:

(*a*) $(1, 1, 1)$, $(3, -1, 4)$, $(-1, 2, 1)$, $(0, 1, 0)$.

(*b*) $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

(*c*) $(0, 0, 0)$, $(-1, 1, 0)$, $(1, 0, 2)$, $(0, 1, -1)$.

(*d*) $(0, 0, 0)$, $(0, 0, 4)$, $(3, 3, 3)$, $(0, 4, 0)$.

**6.** Find the center and radius of the sphere that is the locus of the points three times as far from the point $(a, b, c)$ as from the origin.

**7.** Show that the locus of the points, the ratio of whose distances from two given points is constant, is a sphere except when the ratio is unity.

**8.** Find the positions of the following points relative to the sphere $x^2+y^2+z^2-4x+4y-2z=0$; (*a*) the origin, (*b*) $(2, -2, 1)$, (*c*) $(1, 1, 1)$, (*d*) $(3, -2, 1)$.

**9.** Find the positions of the following planes relative to the sphere

$$x^2+y^2+z^2+4x-3y+6z+5=0:$$

(*a*) $4x+2y+z+2=0$, (*b*) $8x-y-4z+5=0$.

**10.** Find the positions of the following lines relative to the sphere of Ex. 9:

(*a*) $2x-y+2z+7=0,\quad 3x-y-z-10=0.$

(*b*) $3x+8y+z-9=0,\quad x-8y+z+11=0.$

**245. Equations of a Circle.** In solid analytic geometry a curve is represented by two simultaneous equations (§ 221), that is, by the equations of any two surfaces intersecting in the curve. Thus two *linear* equations represent together the line of intersection of the two planes represented by the two equations taken separately (§§ 233, 237).

A linear equation together with the equation of a sphere,

$$\begin{aligned} &Ax+By+Cz+D=0,\\ &x^2+y^2+z^2+ax+by+cz+d=0, \end{aligned} \tag{4}$$

represents the locus of all those points, and only those points, which the plane and sphere have in common. Thus, if the plane intersects the sphere, these simultaneous equations represent the *circle* in which the plane cuts the sphere; if the plane is tangent to the sphere, the equations represent the point of contact; if the plane does not intersect or touch the sphere, the equations are not satisfied simultaneously by any real point.

**246. Sections Perpendicular to Axes. Projecting Cylinders.** In particular, the simultaneous equations

$$z=k,\qquad x^2+y^2+z^2=r^2 \tag{5}$$

represent, if $k<r$, a *circle about the axis Oz* (*i.e.* a circle whose center lies on *Oz* and whose plane is perpendicular to *Oz*). If the value of $z$ obtained from the linear equation be

substituted in the equation of the sphere, we obtain an equation in $x$ and $y$, $x^2 + y^2 = r^2 - k^2$, which represents (since $z$ is arbitrary) the circular cylinder, about $Oz$ as axis, which projects the circle (5) on the plane $Oxy$. Interpreted in the plane $Oxy$, *i.e.* taken together with $z = 0$, this equation represents the projection of the circle (5) on the plane $Oxy$.

Similarly if we eliminate $x$ or $y$ or $z$ between the equations (4), we obtain an equation in $y$ and $z$, $z$ and $x$, or $x$ and $y$, representing the cylinder that projects the circle (4) on the plane $Oyz$, $Ozx$, or $Oxy$, respectively.

**247. Tangent Plane.** The tangent plane to a sphere at any point $P_1$ of the sphere is the plane through $P_1$, at right angles to the radius through $P_1$.

For a sphere whose center is at the origin, $x^2 + y^2 + z^2 = r^2$, the equation of the tangent plane at $P_1(x_1, y_1, z_1)$ is found by observing that its distance from the origin is $r$ and that the direction cosines of its normal are those of $OP_1$, viz. $x_1/r$, $y_1/r$, $z_1/r$. Hence the equation

$$x_1x + y_1y + z_1z = r^2. \tag{6}$$

If the equation of the sphere is given in the general form

$$A(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0,$$

we obtain by transforming to parallel axes through the center the equation

$$x^2 + y^2 + z^2 = \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A} = r^2;$$

the tangent plane at $P_1(x_1, y_1, z_1)$ then is

$$x_1x + y_1y + z_1z = \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A}.$$

Transforming back to the original axes, we have:

$$\left(x_1 + \frac{G}{A}\right)\left(x + \frac{G}{A}\right) + \left(y_1 + \frac{H}{A}\right)\left(y + \frac{H}{A}\right) + \left(z_1 + \frac{I}{A}\right)\left(z + \frac{I}{A}\right)$$
$$= \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A}.$$

Multiplying out and rearranging, we find that *the equation of the tangent plane to the sphere*

$$A(x^2+y^2+z^2)+2\,Gx+2\,Hy+2\,Iz+J=0$$

*at the point* $P_1(x_1,\,y_1,\,z_1)$ is

$$A(x_1x+y_1y+z_1z)+G(x_1+x)+H(y_1+y)+I(z_1+z)+J=0.$$

**248. Intersection of Line and Sphere.** The intersections of a sphere about the origin,

$$x^2+y^2+z^2=r^2,$$

with a line determined by two of its points $P_1(x_1,\,y_1,\,z_1)$ and $P_2(x_2,\,y_2,\,z_2)$, and given in the parameter form [(6), § 239]

$$x=x_1+k(x_2-x_1),\quad y=y_1+k(y_2-y_1),\quad z=z_1+k(z_2-z_1),$$

are found by substituting these values of $x$, $y$, $z$ in the equation of the sphere and solving the resulting quadratic equation in $k$:

$$[x_1+k(x_2-x_1)]^2+[y_1+k(y_2-y_1)]^2+[z_1+k(z_2-z_1)]^2=r^2,$$

which takes the form

$$[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]k^2+2[x_1(x_2-x_1)+y_1(y_2-y_1)$$
$$+z_1(z_2-z_1)]k+(x_1^2+y_1^2+z_1^2-r^2)=0.$$

The line $P_1P_2$ will intersect the sphere in two different points, be tangent to the sphere, or not meet it at all, according as the roots of this equation in $k$ are real and different, real and equal, or imaginary; *i.e.* according as

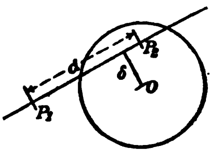

FIG. 123

$$[x_1(x_2-x_1)+y_1(y_2-y_1)+z_1(z_2-z_1)]^2-d^2(x_1^2+y_1^2+z_1^2)+d^2r^2 \overset{>}{\underset{<}{=}} 0,$$

where $d$ denotes the distance of the points $P_1$ and $P_2$. Dividing by $d^2$, we can write this condition in the form

$$r^2-\left[x_1^2+y_1^2+z_1^2-\left(x_1\frac{x_2-x_1}{d}+y_1\frac{y_2-y_1}{d}+z_1\frac{z_2-z_1}{d}\right)^2\right] \overset{>}{\underset{<}{=}} 0,$$

where by § 239 the quantity in square brackets is the square of the distance $\delta$ from the line $P_1P_2$ to the origin $O$ (Fig. 123).

Our condition means therefore that the line $P_1P_2$ meets the sphere in two different points, touches it, or does not meet it at all according as $r > \delta$, $r = \delta$, $r < \delta$, which is obvious geometrically.

**249. Tangent Cone.** The condition for the line $P_1P_2$ to be tangent to the sphere is (§ 244):

$$[x_1(x_2-x_1)+y_1(y_2-y_1)+z_1(z_2-z_1)]^2 =$$
$$(x_1^2+y_1^2+z_1^2-r^2x_2)[(-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2].$$

To give this expression a more symmetric form, let us put, to abbreviate,

$$x_1x_2+y_1y_2+z_1z_2=p, \qquad x_1^2+y_1^2+z_1^2=q_1, \qquad x_2^2+y_2^2+z_2^2=q_2,$$

so that the condition is

$$(p-q_1)^2=(q_1-r^2)(q_1-2p+q_2),$$

*i.e.*

$$p^2-2r^2p=q_1q_2-r^2q_1-r^2q_2;$$

adding $r^4$ in both members, we have

$$(p-r^2)^2=(q_1-r^2)(q_2-r^2),$$

*i.e.*

$$(x_1x_2+y_1y_2+z_1z_2-r^2)^2=(x_1^2+y_1^2+z_1^2-r^2)(x_2^2+y_2^2+z_2^2-r^2).$$

Now keeping the sphere and the point $P_1$ fixed, let $P_2$ vary subject only to this condition, *i.e.* to the condition that $P_1P_2$ shall be tangent to the sphere; the point $P_2$, which we shall now call $P(x, y, z)$ is then any point of the cone of vertex $P_1$ tangent to the sphere. Hence *the equation of the cone of vertex* $P_1(x_1, y_1, z_1)$ *tangent to the sphere* $x^2+y^2+z^2=r^2$ is

$$(x_1^2+y_1^2+z_1^2-r^2)(x^2+y^2+z^2-r^2)=(x_1x+y_1y+z_1z-r^2)^2.$$

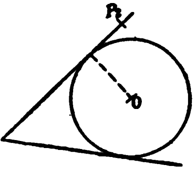

FIG. 124

If, in particular, the point $P_1$ is taken on the sphere so that $x_1^2+y_1^2+z_1^2=r^2$, the equation of the tangent cone reduces to the form $x_1x+y_1y+z_1z=r^2$, which represents the tangent plane at $P_1$.

**250. Inversion.** A sphere of center $O$ and radius $a$ being given, we can find to every point $P$ of space (excepting $O$) one and only one point $P'$ on $OP$ (produced if necessary) such that $OP \cdot OP' = a^2$. The points $P$, $P'$ are said to be *inverse* to each other with respect to the sphere (compare § 56).

Taking rectangular axes through $O$, we find as the relations between the coordinates of the two inverse points $P(x, y, z)$ and $P'(x', y', z')$ if we put $OP = r = \sqrt{x^2 + y^2 + z^2}$, $OP' = r' = \sqrt{x'^2 + y'^2 + z'^2}$:

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{r'}{r} = \frac{rr'}{r^2} = \frac{a^2}{r^2};$$

hence $x' = \dfrac{a^2x}{x^2 + y^2 + z^2}$, $y' = \dfrac{a^2y}{x^2 + y^2 + z^2}$, $z' = \dfrac{a^2z}{x^2 + y^2 + z^2}$;

and similarly

$$x = \frac{a^2x'}{x'^2 + y'^2 + z'^2}, \quad y = \frac{a^2y'}{x'^2 + y'^2 + z'^2}, \quad z = \frac{a^2z'}{x'^2 + y'^2 + z'^2}.$$

These equations enable us to find to any surface whose equation is given the equation of the inverse surface, by simply substituting for $x$, $y$, $z$ their values.

Thus it can be shown, that by inversion every sphere is transformed into a sphere or a plane. The proof is similar to the corresponding proposition in plane analytic geometry (§ 57) and is left as an exercise.

## EXERCISES

1. Find the radius of the circle which is the intersection: (*a*) of the plane $y = 6$ with the sphere $x^2 + y^2 + z^2 - 6y = 0$; (*b*) of the plane $2x - 3y + z - 2 = 0$ with the sphere $x^2 + y^2 + z^2 - 6x + 2y - 15 = 0$.

2. A line perpendicular to the plane of a circle through its center is called the *axis* of the circle. Find the circle: (*a*) which lies in the plane $z = 4$, has a radius 3 and $Oz$ as axis; (*b*) which lies in the plane $y = 5$, has a radius 2 and the line $x - 3 = 0$, $z - 4 = 0$ as axis.

3. Find the circles of radius 3 on the sphere of radius 4 about the origin whose common axis is equally inclined to the coordinate axes.

4. Does the line joining the points $(2, -1, -6)$, $(-1, 2, 3)$ intersect the sphere $x^2 + y^2 + z^2 = 10$? Find the points of intersection.

**5.** Find the planes tangent to the following spheres at the given points:

(*a*) $x^2+y^2+z^2-3y-5z-2=0$, at $(2, -1, 3)$;
(*b*) $x^2+y^2+z^2+2x-6y+z-1=0$, at $(0, 1, -3)$;
(*c*) $3(x^2+y^2+z^2)-5x+2y-z=0$, at the origin;
(*d*) $x^2+y^2+z^2-ax-by-cz=0$, at $(a, b, c)$.

**6.** Find the tangent cone: (*a*) from $(4, 1, -2)$ to $x^2+y^2+z^2=9$; (*b*) from $(2a, 0, 0)$ to $x^2+y^2+z^2=a^2$; (*c*) from $(4, 4, 4)$ to $x^2+y^2+z^2=16$; (*d*) from $(1, -5, 3)$ to $x^2+y^2+z^2=9$.

**7.** Find the cone with vertex at the origin tangent to the sphere $(x-2a)^2+y^2+z^2=a^2$.

**8.** Show that, by inversion with respect to the sphere $x^2+y^2+z^2=a^2$, every plane (except one through the center) is transformed into a sphere passing through the origin.

**9.** With respect to the sphere $x^2+y^2+z^2=25$, find the surfaces inverse to (*a*) $x=5$, (*b*) $x-y=0$, (*c*) $4(x^2+y^2+z^2)-20x-25=0$.

**10.** Show that by inversion with respect to the sphere $x^2+y^2+z^2=a^2$ every line through the origin is transformed into itself.

**11.** With respect to the sphere $x^2+y^2+z^2=a^2$, find the surface inverse to the plane tangent at the point $P_1(x_1, y_1, z_1)$.

**12.** Show that all spheres with center at the center of inversion are transformed into concentric spheres by inversion.

**13.** What is the curve inverse to the circle $x^2+y^2+z^2=25$, $z=4$, with respect to the sphere $x^2+y^2+z^2=16$?

**251. Poles and Polars.** Let $P$ and $P'$ be inverse points with respect to a given sphere; then the plane $\pi$ through $P'$, at right angles to $OP$ ($O$ being the center of the sphere), is called the ***polar plane*** of the point $P$, and $P$ is called the ***pole*** of the plane $\pi$, with respect to the sphere.

*With respect to a sphere of radius a, with center at the origin,*

$$x^2+y^2+z^2=a^2,$$

the *equation of the polar plane of any point* $P_1(x_1, y_1, z_1)$ is readily found by observing that its distance from the origin is $a^2/r_1$, and that the direction cosines of its normal are equal to $x_1/r_1$, $y_1/r_1$, $z_1/r_1$, where $r_1^2 = x_1^2 + y_1^2 + z_1^2$; the equation is therefore

$$x_1x + y_1y + z_1z = a^2.$$

If, in particular, the point $P_1$ lies on the sphere, this equation, by § 254 (6), represents the tangent plane at $P_1$. Hence the polar plane of any point of the sphere is the tangent plane at that point; this also follows from the definition of the polar plane.

**252.** With respect to the same sphere the polar planes of any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are

$$x_1x + y_1y + z_1z = a^2 \quad \text{and} \quad x_1x + y_2y + z_2z = a^2.$$

Now the condition for the polar plane of $P_1$ to pass through $P_2$ is

$$x_1x_2 + y_1y_2 + z_1z_2 = a^2;$$

but this is also the condition for the polar plane of $P_2$ to pass through $P_1$. Hence *the polar planes of all the points of any plane* $\pi$ (not passing through the origin $O$) *pass through a common point, namely, the pole of the plane* $\pi$; and conversely, *the poles of all the planes through a common point P lie in a plane, namely, the polar plane of P.*

**253.** The polar plane of any point $P$ of the line determined by two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ (always with respect to the same sphere $x^2 + y^2 + z^2 = a^2$) is

$$[x_1 + k(x_2 - x_1)]x + [y_1 + k(y_2 - y_1)]y + [z_1 + k(z_2 - z_1)]z = a^2.$$

This equation can be written in the form

$$x_1x + y_1y + z_1z - a^2 + \frac{k}{1-k}(x_2x + y_2y + z_2z - a^2) = 0,$$

which for a variable $k$ represents the planes of the pencil whose axis is the intersection of the polar planes of $P_1$ and $P_2$. Hence *the polar planes of all the points of a line* $\lambda$ *pass through a common line;* and conversely, *the poles of all the planes of a pencil lie on a line.*

Two lines related in this way are called ***conjugate lines*** (or conjugate axes, reciprocal polars). Thus the line $P_1P_2$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

and the line
$$x_1x + y_1y + z_1z = a^2,$$
$$x_2x + y_2y + z_2z = a^2$$
are conjugate with respect to the sphere $x^2 + y^2 + z^2 = a^2$.

As the direction cosines of these lines are proportional to
$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1$$
and
$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix},$$
respectively, the two conjugate lines are at right angles (§ 236).

**254.** By the method used in the corresponding problem in the plane (§ 60) it can be shown that the polar plane of any point $P_1(x_1, y_1, z_1)$ with respect to any sphere
$$A(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0$$
is
$$A(x_1x + y_1y + z_1z) + G(x_1 + x) + H(y_1 + y) + I(z_1 + z) + J = 0.$$

**255. Power of a Point.** If in the left-hand member of the equation of the sphere
$$(x - h)^2 + (y - j)^2 + (z - k)^2 - r^2 = 0$$
we substitute for $x$, $y$, $z$ the coordinates $x_1, y_1, z_1$ of any point *not* on the sphere, we obtain an expression $(x_1 - h)^2 + (y_1 - j)^2 + (z_1 - k)^2 - r^2$ different from zero which is called the ***power of the point*** $P_1(x_1, y_1, z_1)$ ***with respect to the sphere.***

As $(x_1 - h)^2 + (y_1 - j)^2 + (z_1 - k)^2$ is the square of the distance $d$ between the point $P_1$ and the center $C$ of the sphere, we can write the power of $P_1$ briefly
$$d^2 - r^2;$$
the power of $P_1$ is positive or negative according as $P_1$ lies outside or within the sphere. For a point $P_1$ outside, the power is evidently the square of the length of a tangent drawn from $P_1$ to the sphere.

**256. Radical Plane, Axis, Center.** The locus of a point whose powers with respect to the two spheres
$$x^2 + y^2 + z^2 + a_1x + b_1y + c_1z + d_1 = 0,$$
$$x^2 + y^2 + z^2 + a_2x + b_2y + c_2z + d_2 = 0$$
are equal is evidently the plane
$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z + d_1 - d_2 = 0,$$
which is called the ***radical plane*** of the two spheres. It always exists unless the two spheres are concentric.

It is easily proved that the three radical planes of any three spheres (no two of which are concentric) are planes of the same pencil (§ 228); and hence that the locus of the points of equal power with respect to three spheres is a straight line. This line is called the ***radical axis*** of the three spheres; it exists unless the centers lie in a straight line.

The six radical planes of four spheres, taken in pairs, are in general planes of a sheaf (§ 229). Hence there is in general but one point of equal power with respect to four spheres. This point, the ***radical center*** of the four spheres, exists unless the four centers lie in a plane.

### 257. Family of Spheres.

The equation

$$(x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1)+k(x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2)=0$$

represents a ***family***, or ***pencil, of spheres***, provided $k \neq -1$. If the two spheres

$$x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1=0,$$
$$x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2=0$$

intersect, every sphere of the pencil passes through the common circle of these two spheres. If $k=-1$, the equation represents the radical plane of the two spheres.

#### EXERCISES

**1.** Find the radius of the circle in which the polar plane of the point $(4, 3, -1)$ with respect to $x^2+y^2+z^2=16$ cuts the sphere.

**2.** Find the radius of the circle in which the polar plane of the point $(5, -1, 2)$ with respect to $x^2+y^2+z^2-2x+4y=0$ cuts the sphere.

**3.** Show that the plane $3x+y-4z=19$ is tangent to the sphere $x^2+y^2+z^2-2x-4y-6z-12=0$, and find the point of contact.

**4.** If a point describes the plane $4x-5y-3x=16$, find the coordinates of that point about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=16$.

**5.** If a point describes the plane $2x+3y+z=4$, find that point about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=8$.

**6.** If a point describes the line $\dfrac{x-2}{3}=\dfrac{y+3}{5}=\dfrac{z-1}{-2}$, find the equations of that line about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=25$. Show that the two lines are perpendicular.

**7.** If a point describe the line $2x-3y+4z=2$, $x+y+z=3$, find the equations of that line about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=16$. Show that the two lines are perpendicular.

**8.** Find the sphere through the origin that passes through the circle of intersection of the spheres $x^2+y^2+z^2-3x+4y-5z-8=0$, $x^2+y^2+z^2-2x+y-z-10=0$.

**9.** Show that the locus of a point whose powers with respect to two given spheres have a constant ratio is a sphere except when the ratio is unity.

**10.** Show that the radical plane of two spheres is perpendicular to the line joining their centers.

**11.** Show that the radical plane of two spheres tangent internally or externally is their common tangent plane.

**12.** Find the equations of the radical axis of the spheres $x^2+y^2+z^2-3x-2y-z-4=0$, $x^2+y^2+z^2+5x-3y-2z-8=0$, $x^2+y^2+z^2-16=0$.

**13.** Find the radical center of the spheres $x^2+y^2+z^2-6x+2y-z+6=0$, $x^2+y^2+z^2-10=0$, $x^2+y^2+z^2+2x-3y+5z-6=0$, $x^2+y^2+z^2-2x+4y-12=0$.

**14.** Show that the three radical planes of three spheres are planes of the same pencil.

**15.** Two spheres are said to be orthogonal when their tangent planes at every point of their circle of intersection are perpendicular. Show that the two spheres $x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1=0$, $x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2=0$ are orthogonal when $a_1a_2+b_1b_2+c_1c_2=2(d_1+d_2)$.

**16.** Write the equation of the cone tangent to the sphere $x^2+y^2+z^2=r^2$ with vertex $(0, 0, z_1)$. Divide this equation by $z_1^2$ and let the vertex recede indefinitely, *i.e.* let $z_1$ increase indefinitely. The equation $x^2+y^2=r^2$, thus obtained, represents the cylinder with axis along the axis $Oz$ and tangent to the sphere $x^2+y^2+z^2=r^2$.

# CHAPTER XIII

## QUADRIC SURFACES

**253. The Ellipsoid.** The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is called an ***ellipsoid.*** Its shape is best investigated by taking cross-sections at right angles to the axes of coordinates.

Thus the coordinate plane $Oyz$ whose equation is $x=0$ intersects the ellipsoid in the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Any other plane perpendicular to the axis $Ox$ (Fig. 125) at the distance $h<a$ from the plane $Oyz$ intersects the ellipsoid in an ellipse whose equation is

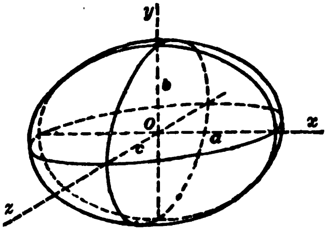

FIG. 125

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{h^2}{a^2},$$

*i.e.*
$$\frac{y^2}{b^2\left(1-\dfrac{h^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\dfrac{h^2}{a^2}\right)}=1$$

Strictly speaking this is the equation of the cylinder that projects the cross-section on the plane *Oyz*. But it can also be interpreted as the equation of the cross-section itself, referred to the point $(h, 0, 0)$ as origin and axes in the cross-section parallel to *Oy* and *Oz*.

Notice that as $h < a$, $h^2/a^2$, and hence also $1 - h^2/a^2$, is a positive proper fraction. The semi-axes $b\sqrt{1 - h^2/a^2}$, $c\sqrt{1 - h^2/a^2}$ of the cross-section are therefore less than $b$ and $c$, respectively. As $h$ increases from 0 to $a$, these semi-axes gradually diminish from $b$, $c$ to 0.

**259. Cross-Sections.** Cross-sections on the opposite side of the plane *Oyz* give the same results; the ellipsoid is evidently symmetric with respect to the plane *Oyz*.

By the same method we find that cross-sections perpendicular to the axes *Oy* and *Oz* give ellipses with semi-axes diminishing as we recede from the origin. The surface is evidently symmetric to each of the coordinate planes. It follows that the origin is a ***center***, *i.e.* every chord through that point is bisected at that point. In other words, if $(x, y, z)$ is a point of the surface, so is $(-x, -y, -z)$. Indeed, it is clear from the equation that if $(x, y, z)$ lies on the ellipsoid, so do the seven other points $(x, y, -z)$, $(x, -y, z)$, $(-x, y, z)$, $(x, -y, -z)$, $(-x, y, -z)$, $(-x, -y, z)$, $(-x, -y, -z)$. A chord through the center is called a ***diameter***.

It follows that it suffices to study the shape of the portion of the surface contained in one octant, say that contained in the trihedral formed by the positive axes *Ox*, *Oy*, *Oz*; the remaining portions are then obtained by reflection in the coordinate planes.

The ellipsoid is a *closed surface;* it does not extend to infinity; indeed it is completely contained within the parallelepiped with center at the origin and edges $2a$, $2b$, $2c$, parallel to *Ox*, *Oy*, *Oz*, respectively.

**260. Special Cases.** In general, the *semi-axes* $a$, $b$, $c$ of the ellipsoid, *i.e.* the intercepts made by it on the axes of coordinates, are different. But it may happen that two of them, or even all three, are equal.

In the latter case, *i.e.* if $a = b = c$, the ellipsoid evidently reduces to a ***sphere***.

If two of the axes are equal, *e.g.* if $b = c$, the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

is called an ***ellipsoid of revolution*** because it can be generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the axis $Ox$ (Fig. 126). Any cross-section at right angles to $Ox$, the ***axis of revolution***, is a circle, while the cross-sections at right angles to $Oy$ and $Oz$ are ellipses. The circular cross-section in the plane $Oyz$ is called the ***equator***; the intersections of the surface with the axis of revolution are the ***poles***.

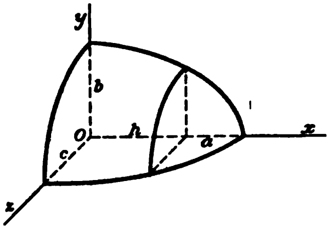

FIG. 126

If $a > b$ ($a$ being the intercept on the axis of revolution), the ellipsoid of revolution is called ***prolate***; if $a < b$, it is called ***oblate***. In astronomy the ellipsoid of revolution is often called ***spheroid***, the surfaces of the planets which are approximately ellipsoids of revolution being nearly spherical. Thus for the surface of the earth the major semi-axis, *i.e.* the radius of the equator, is 3962.8 miles while the minor semi-axis, *i.e.* the distance from the center to the north or south pole, is 3949.6 miles.

**261. Surfaces of Revolution.** A surface that can be generated by the revolution of a plane curve about a line in the plane of the curve is called a ***surface of revolution.*** Any such surface is fully determined by the generating curve and the position of the axis of revolution with respect to the curve.

Let us take the axis of revolution as axis *Ox*, and let the equation of the generating curve be

$$y = f(x).$$

As this curve revolves about *Ox*, any point *P* of the curve (Fig. 127) describes a circle about *Ox* as axis, with a radius equal to the ordinate $f(x)$ of the generating curve. For any position of *P* we have therefore

$$y^2 + z^2 = [f(x)]^2,$$

and this is the *equation of the surface of revolution.*

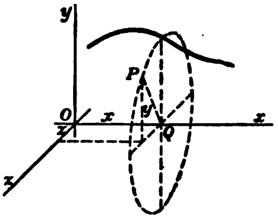

FIG. 127

Thus if the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

revolves about the axis *Ox*, we find since $y = \pm (b/a)\sqrt{a^2 - x^2}$ for the ellipsoid of revolution so generated the equation

$$y^2 + z^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

which agrees with that of § 260.

Any section of a surface of revolution at right angles to the axis of revolution is of course a circle; these sections are called ***parallel circles,*** or simply *parallels* (as on the earth's surface). Any section of a surface of revolution by a plane passing through the axis of revolution is called a ***meridian section;*** it consists of the generating curve and its reflection in the axis of revolution.

## EXERCISES

**1.** An ellipsoid has six *foci*, viz. the foci of the three ellipses in which the ellipsoid is intersected by its planes of symmetry. Determine the coordinates of these foci: (*a*) for an ellipsoid with semi-axes 1, 2, 3; (*b*) for the earth (see § 260); (*c*) for an ellipsoid of semi-axes 10, 8, 1; (*d*) for an ellipsoid of semi-axes 1, 1, 5.

**2.** Find the equations of the surfaces of revolution generated by revolving the following curves about the given lines:

(*a*) $y = x^2$, about the axis $Ox$.
(*b*) $y^2 = x$, about the latus rectum.
(*c*) $x^2 + y^2 - 2x = 0$, about the axis $Oy$.
(*d*) $x^2 - y^2 = 1$, about the axis $Ox$.

**3.** Find the equation of the paraboloid of revolution generated by the revolution of the parabola $y^2 = 4ax$ about $Ox$.

**4.** Find the equation of a *torus*, or *anchor-ring*, *i.e.* the surface generated by the revolution of a circle of radius $a$ about a line in its plane at the distance $b > a$ from its center.

**5.** Find the equation of the surface generated by the revolution of a circle of radius $a$ about a line in its plane at the distance $b < a$ from its center. Is the appearance of this surface noticeably different from the surface of Ex. 4? What happens to this surface when $b = 0$; when $b = a$?

**6.** Find the equation of the surface generated by the revolution of the parabola $y^2 = 4ax$ about: (*a*) the tangent at the vertex; (*b*) the latus rectum.

**7.** Find the equation of the surface generated by the revolution of the hyperbola $xy = a^2$ about an asymptote.

**8.** Find the cone generated by the revolution of the line $y = mx + b$ about: (*a*) $Ox$, (*b*) $Oy$.

**9.** How are the following surfaces of revolution generated?

(*a*) $y^2+z^2=x^4$. (*b*) $2x^2+2y^2-3z=0$. (*c*) $x^2+y^2-z^2-2x+4=0$.

**10.** Find the equation of the surface generated by the revolution of the ellipse $x^2 + 4y^2 - 4x = 0$: (*a*) about the major axis; (*b*) about the minor axis; (*c*) about the tangent at the origin.

**262. Hyperboloid of One Sheet.** The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is called a ***hyperboloid of one sheet*** (Fig. 128). The intercepts on the axes $Ox$, $Oy$ are $\pm a$, $\pm b$; the axis $Oz$ does not intersect the surface.

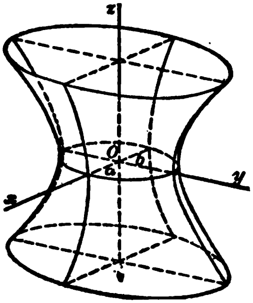

FIG. 128

**263. Cross-Sections.** The plane $Oxy$ intersects the surface in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

cross-sections perpendicular to $Oz$ give ellipses with ever-increasing semi-axes.

The planes $Oyz$ and $Ozx$ intersect the surface in the hyperbolas

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1.$$

Any plane perpendicular to $Ox$, at the distance $h$ from the origin, intersects the hyperboloid in a hyperbola, viz.

$$\frac{y^2}{b^2\left(1-\dfrac{h^2}{a^2}\right)}-\frac{z^2}{c^2\left(1-\dfrac{h^2}{a^2}\right)}=1.$$

As long as $h < a$ this hyperbola has its transverse axis parallel to $Oy$ while for $h > a$ the transverse axis is parallel to $Oz$; for $h = a$ the equation reduces to $y^2/b^2 - z^2/c^2 = 0$ and represents two straight lines, viz. the parallels through $(a, 0, 0)$ to the asymptotes of the hyperbola $y^2/b^2 - z^2/c^2 = 1$ which is the intersection of the surface with the plane $Oyz$.

Similar considerations apply to the cross-sections perpendicular to $Oy$.

The hyperboloid has the same properties of symmetry as the ellipsoid (§ 259); the origin is a *center*, and it suffices to investigate the shape of the surface in one octant.

**264. Hyperboloid of Revolution of One Sheet.** If in the hyperboloid of one sheet we have $a = b$, the cross-sections perpendicular to the axis $Oz$ are all circles so that the surface can be generated by the revolution of the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

about $Oz$. Such a surface is called a ***hyperboloid of revolution of one sheet.***

**265. Other Forms.** The equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

also represent hyperboloids of one sheet which can be investigated as in §§ 262–264. In the former of these the axis $Oy$, in the latter the axis $Ox$, does not meet the surface.

Every hyperboloid of one sheet extends to infinity.

**266. Hyperboloid of Two Sheets.** The surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called a ***hyperboloid of two sheets*** (Fig. 129).

The intercepts on $Ox$ are ± $a$; the axes $Oy$, $Oz$ do not meet the surface.

**267. Cross-Sections.** The cross-sections at right angles to $Ox$, at the distance $h$ from the origin are

$$-\frac{y^2}{b^2\left(1-\frac{h^2}{a^2}\right)}-\frac{z^2}{c^2\left(1-\frac{h^2}{a^2}\right)}=1;$$

these are imaginary as long as $h < a$; for $h > a$ they are ellipses with ever-increasing semi-axes as we recede from the origin.

The cross-sections at right angles to $Oy$ and $Oz$ are hyperbolas.

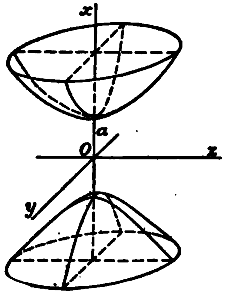

FIG. 129

The hyperboloid of two sheets, like that of one sheet and like the ellipsoid, has three mutually rectangular planes of symmetry whose intersection is therefore a *center*.

The surfaces

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are hyperboloids of two sheets, the former being met by $Oy$, the latter by $Oz$, in real points.

The hyperboloid of two sheets extends to infinity.

**268. Hyperboloid of Revolution of Two Sheets.** If $b = c$ in the equation of § 266, the cross-sections at right angles to $Ox$ are circles and the surface becomes a *hyperboloid of revolution of two sheets.*

**269. Imaginary Ellipsoid.** The equation

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is not satisfied by any point with real coordinates. It is sometimes said to represent an *imaginary ellipsoid.*

**270. The Paraboloids.** The surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz,$$

which are called the ***elliptic paraboloid*** (Fig. 130) and ***hyperbolic paraboloid*** (Fig. 131), respectively, have each only two planes of symmetry, viz. the planes $Oyz$ and $Ozx$. We here assume that $c \neq 0$. The cross-sections at right angles to the axis $Oz$ are evidently ellipses in the case of the elliptic paraboloid, and hyperbolas in the case of the hyperbolic paraboloid. The plane $Oxy$ itself has only the origin in common with the elliptic paraboloid; it intersects the hyperbolic paraboloid in the two lines $x^2/a^2 - y^2/b^2 = 0$, *i.e.* $y = \pm\, bx/a$.

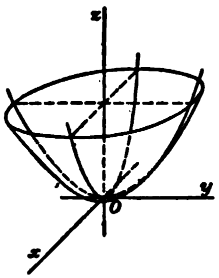

FIG. 130

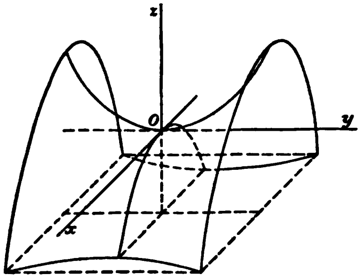

FIG. 131

The intersections of the elliptic paraboloid (Fig. 130) with the planes $Oyz$ and $Ozx$ are parabolas with $Oz$ as axis and $O$ as vertex, opening in the sense of positive $z$ if $c$ is positive, in the sense of negative $z$ if $c$ is negative. Planes parallel to these coordinate planes intersect the elliptic paraboloid in parabolas with axes parallel to $Oz$, but with vertices not on the axes $Ox$, $Oy$, respectively.

For the hyperbolic paraboloid (Fig. 131), which is saddle-shaped at the origin, the intersections with the planes $Oyz$ and

*Ozx* are also parabolas with *Oz* as axis; if $c$ is positive the parabola in the plane *Oyz* opens in the sense of negative $z$, that in the plane *Ozx* opens in the sense of positive $z$. Similarly for the parallel sections.

**271. Paraboloid of Revolution.** If in the equation of the elliptic paraboloid we have $a = b$, it reduces to the form

$$x^2 + y^2 = 2\,pz.$$

This represents a surface of revolution, called the ***paraboloid of revolution.*** This surface can be regarded as generated by the revolution of the parabola $y^2 = 2\,pz$ about the axis *Oz*.

**272. Elliptic Cone.** The surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is an ***elliptic cone***, with the origin as vertex and the axis *Oz* as axis (Fig. 132).

The plane *Oxy* has only the origin in common with the surface. Every parallel plane $z = k$, whether $k$ be positive or negative, intersects the surface in an ellipse, with semi-axes increasing proportionally to $k$.

The plane *Oyz*, as well as the plane *Ozx*, intersects the surface in two straight lines through the origin. Every plane parallel to *Oyx* or to *Ozx* intersects the surface in a hyperbola.

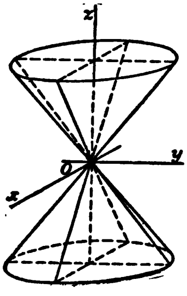

FIG. 132

**273. Circular Cone.** If in the equation of the elliptic cone we have $a = b$, the cross-sections at right angles to the axis *Oz* become circles. The cone is then an ordinary ***circular cone***, or

***cone of revolution***, which can be generated by the revolution of the line $y = (a/c)z$ about the axis $Oz$. Putting $a/c = m$ we can write the equation of a cone of revolution about $Oz$, with vertex at $O$, in the form

$$x^2 + y^2 = m^2z^2.$$

**274. Quadric Surfaces.** The ellipsoid, the two hyperboloids, the two paraboloids, and the elliptic cone are called ***quadric surfaces*** because their cartesian equations are all of the second degree.

Let us now try to determine, conversely, all the various loci that can be represented by the ***general equation of the second degree***

$$Ax^2 + By^2 + Cz^2 + 2\,Dyz + 2\,Ezx + 2\,Fxy \\ + 2\,Gx + 2\,Hy + 2\,Iz + J = 0.$$

In studying the equation of the second degree in $x$ and $y$ (§ 253) it was shown that the term in $xy$ can always be removed by turning the axes about the origin through a certain angle. Similarly, it can be shown in the case of three variables that by a properly selected rotation of the coordinate trihedral about the origin the terms in $yx$, $zx$, $xy$ can in general all be removed so that the equation reduces to the form

$$(1) \qquad Ax^2 + By^2 + Cz^2 + 2\,Gx + 2\,Hy + 2\,Iz + J = 0.$$

This transformation being somewhat long will not be given here. We shall proceed to classify the surfaces represented by equations of the form (1).

**275. Classification.** The equation (1) can be further simplified by completing the squares. *Three cases* may be distinguished according as the coefficients $A$, $B$, $C$ are all three different from zero, one only is zero, or two are zero.

CASE (*a*): $A \neq 0, B \neq 0, C \neq 0$. Completing the squares in $x, y, z$ we find

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{H}{B}\right)^2+C\left(z+\frac{I}{C}\right)^2=\frac{G^2}{A}+\frac{H^2}{B}+\frac{I^2}{C}-J=J_1.$$

Referred to parallel axes through the point $(-G/A, -H/B, -I/C)$ this equation becomes

$$Ax^2+By^2+Cz^2=J_1. \tag{2}$$

CASE (*b*): $A \neq 0, B \neq 0, C=0$. Completing the squares in $x$ and $y$ we find

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{H}{B}\right)^2+2\,Iz=\frac{G^2}{A}+\frac{H^2}{B}-J=J_2.$$

If $I \neq 0$, we can transform to parallel axes through the point $(-G/A, -H/B, J_2/2\,I)$ so that the equation becomes

$$Ax^2+By^2+2\,Iz=0. \tag{3}$$

If, however, $I=0$, we obtain by transforming to the point $(-G/A, -H/B, 0)$

$$Ax^2+By^2=J_2. \tag{3'}$$

CASE (*c*): $A \neq 0, B=0, C=0$. Completing the square in $x$ we have

$$A\left(x+\frac{G}{A}\right)^2+2\,Hy+2\,Iz=\frac{G^2}{A}-J=J_3.$$

If $H$ and $I$ are not both zero, we can transform to parallel axes through the point $(-G/A, J_3/2\,H, 0)$ or through $(-G/A, 0, J_3/2\,I)$ and find

$$Ax^2+2\,Hy+2\,Iz=0. \tag{4}$$

If $H=0$ and $I=0$, we transform to the point $(-G/A, 0, 0)$ so that we find .

$$Ax^2=J_3. \tag{4'}$$

**276. Squared Terms all Present. Case (a).** We proceed to discuss the loci represented by (2). If $J_1 \neq 0$, we can divide (2) by $J_1$ and obtain:

($\alpha$) if $A/J_1$, $B/J_1$, $C/J_1$ are positive, an *ellipsoid* (§ 258);

($\beta$) if two of these coefficients are positive while the third is negative, a *hyperboloid of one sheet* (§ 262);

($\gamma$) if one coefficient is positive while two are negative, a *hyperboloid of two sheets* (§ 266);

($\delta$) if all three coefficients are negative, the equation is not satisfied by any real point (§ 269);

If $J_1 = 0$, the equation (2) represents an *elliptic cone* (§ 272) unless $A$, $B$, $C$ all have the same sign, in which case *the origin* is the only point represented.

**277. Case (b).** The equation (3) of § 275 evidently furnishes the two *paraboloids* (§ 270); the paraboloid is *elliptic* if $A$ and $B$ have the same sign; it is *hyperbolic* if $A$ and $B$ are of opposite sign.

The equation (3'), since it does not contain $z$ and hence leaves $z$ arbitrary, represents the *cylinder, with generators parallel to Oz, passing through the conic* $Ax^2 + By^2 = J_2$. As $A$ and $B$ are assumed different from zero, this conic is an ellipse if $A/J_2$ and $B/J_2$ are both positive, a hyperbola if $A/J_2$ and $B/J_2$ are of opposite sign, and it is imaginary if $A/J_2$ and $B/J_2$ are both negative. This assumes $J_2 \neq 0$. If $J_2 = 0$, the conic degenerates into two straight lines, real or imaginary; the cylinder degenerates into two planes if the lines are real.

**278. Case (c).** There remain equations (4) and (4'). To simplify (4) we may turn the coordinate trihedral about $Ox$ through an angle whose tangent is $-H/I$; this is done by putting

$$x = x', \quad y = \frac{Iy' + Hz'}{\sqrt{H^2 + I^2}}, \quad z = \frac{-Hy' + Iz'}{\sqrt{H^2 + I^2}};$$

our equation then becomes

$$Ax'^2 + 2\sqrt{H^2 + I^2}z' = 0.$$

It evidently represents a *parabolic cylinder*, with generators parallel to $Oy$.

Finally, the equation (4') is readily seen to represent *two planes* perpendicular to $Ox$, real or imaginary, unless $J_3 = 0$, in which case it represents the plane $Oyz$.

### EXERCISES

**1.** Name and locate the following surfaces:

(*a*) $x^2 + 2y^2 + 3z^2 = 4.$
(*b*) $x^2 + y^2 - 5z - 6 = 0.$
(*c*) $x^2 - y^2 + z^2 = 4.$
(*d*) $x^2 - y^2 + z^2 + 3z + 6 = 0.$
(*e*) $2y^2 - 4z^2 - 5 = 0.$
(*f*) $2x^2 + y^2 + 3z^2 + 5 = 0.$
(*g*) $5z^2 + 2x^2 = 10.$
(*h*) $z^2 - 9 = 0.$
(*i*) $x^2 - y + 1 = 0.$
(*j*) $x^2 - y^2 - z^2 + 6z = 9.$
(*k*) $x^2 + 3y^2 + z^2 + 4z + 4 = 0.$
(*l*) $z^2 + y - 9 = 0.$

**2.** The cone

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$$

is called the *asymptotic cone* of the hyperboloid of one sheet

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.$$

Show that as $z$ increases the two surfaces approach each other, *i.e.* they bear a relation similar to a hyperbola and its asymptotes.

**3.** What is the asymptotic cone of the hyperboloid of two sheets?

**4.** Show that the intersection of a hyperboloid of two sheets with any plane actually cutting the surface is an ellipse, parabola, or hyperbola. Determine the position of the plane for each conic.

**5.** Show that in general nine points determine a quadric surface and that the equation may be written as a determinant of the tenth order equated to zero.

**6.** Show that the surface inverse to the cylinder $x^2 + y^2 = a^2$, with respect to the sphere $x^2 + y^2 + z^2 = a^2$, is the torus generated by the revolution of the circle $(y - a/2)^2 + z^2 = a^2$ about the axis $Ox$.

**7.** Determine the nature of the surface $xyz = a^3$ by means of cross-sections.

**279. Tangent Plane to the Ellipsoid.** The plane tangent to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

can be found as follows (compare §§ 255, 256). The equations of the line joining any two given points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are

$$x=x_1+k(x_2-x_1),\quad y=y_1+k(y_2-y_1),\quad z=z_1+k(z_2-z_1).$$

This line will be tangent to the ellipsoid if the quadratic in $k$

$$\frac{[x_1+k(x_2-x_1)]^2}{a^2}+\frac{[y_1+k(y_2-y_1)]^2}{b^2}+\frac{[z_1+k(z_2-z_1)]^2}{c^2}=1$$

has equal roots. Writing this quadratic in the form

$$\left[\frac{(x_2-x_1)^2}{a^2}+\frac{(y_2-y_1)^2}{b^2}+\frac{(z_2-z_1)^2}{c^2}\right]k^2$$
$$+2\left[\frac{x_1(x_2-x_1)}{a^2}+\frac{y_1(y_2-y_1)}{b^2}+\frac{z_1(z_2-z_1)}{c^2}\right]k+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0,$$

we find the condition

$$\left[\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}-1\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\right]^2$$
$$=\left[\frac{(x_2-x_1)^2}{a^2}+\frac{(y_2-y_1)^2}{b^2}+\frac{(z_2-z_1)^2}{c^2}\right]\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right).$$

If now we keep the point $(x_1, y_1, z_1)$ fixed, but let the point $(x_2, y_2, z_2)$ vary subject to this condition, it will describe the cone, with vertex $(x_1, y_1, z_1)$, tangent to the ellipsoid; to indicate this we shall drop the subscripts of $x_2$, $y_2$, $z_2$. If, in particular, the point $(x_1, y_1, z_1)$ be chosen on the ellipsoid, we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,$$

and the cone becomes the tangent plane. The *equation of the tangent plane to the ellipsoid at the point* $(x_1, y_1, z_1)$ is, therefore:

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=1.$$

**280. Tangent Planes to Hyperboloids.** In the same way it can be shown that the *tangent planes to the hyperboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

at $(x_1, y_1, z_1)$ are

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-\frac{z_1z}{c^2}=1,\quad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}-\frac{z_1z}{c^2}=1.$$

By an equally elementary, but somewhat longer, calculation it can be shown that the *tangent plane to the quadric surface*

$$Ax^2+By^2+Cz^2+2\,Dyz+2\,Ezx+2\,Fxy$$
$$+2\,Gx+2\,Hy+2\,Iz+J=0$$

at $(x_1, y_1, z_1)$ is:

$$Ax_1x+By_1y+Cz_1z+D(y_1z+z_1y)+E(z_1x+x_1z)+F(x_1y+y_1x)$$
$$+G(x_1+x)+H(y_1+y)+I(z_1+z)+J=0.$$

In particular, the *tangent planes to the paraboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz$$

are

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=c(z_1+z),\quad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}=c(z_1+z).$$

**281. Ruled Surfaces.** A surface that can be generated by the motion of a straight line is called a *ruled surface;* the line is called the *generator.*

The plane is a ruled surface. Among the quadric surfaces not only the cylinders and cones but also the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces.

**282. Rulings on a Hyperboloid of One Sheet.** To show this for the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

we write the equation in the form

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$$

and factor both members:

$$\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right).$$

It is then apparent that any point whose coordinates satisfy the two equations

$$\frac{y}{b}+\frac{z}{c}=k\left(1+\frac{x}{a}\right),\quad \frac{y}{b}-\frac{z}{a}=\frac{1}{k}\left(1-\frac{x}{a}\right),$$

where $k$ is an arbitrary parameter, lies on the hyperboloid. These two equations represent for every value of $k$ ($\neq 0$) a straight line. The hyperboloid of one sheet contains therefore the family of lines represented by the last two equations with variable $k$.

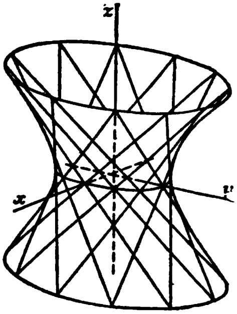

FIG. 133

In exactly the same way it is shown that the same hyperboloid also contains the family of lines

$$\frac{y}{b}-\frac{z}{c}=k'\left(1+\frac{x}{a}\right),\quad \frac{y}{b}+\frac{z}{c}=\frac{1}{k'}\left(1-\frac{x}{a}\right).$$

Thus *every hyperboloid of one sheet contains two sets of rectilinear generators* (Fig. 133).

**283. Rulings on a Hyperbolic Paraboloid.** The hyperbolic paraboloid (Fig. 134)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz$$

also *contains two sets of rectilinear generators*, namely,

$$\frac{x}{a}+\frac{y}{b}=k\cdot 2\,cz,\quad \frac{x}{a}-\frac{y}{b}=\frac{1}{k},$$

and

$$\frac{x}{a}-\frac{y}{b}=k'\cdot 2\,cz,\quad \frac{x}{a}+\frac{y}{b}=\frac{1}{k'}.$$

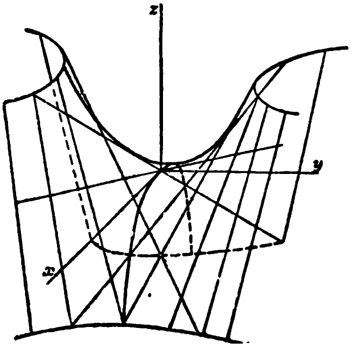

FIG. 134

EXERCISES

1. Derive the equation of the tangent plane to:

(*a*) the elliptic paraboloid; (*b*) the hyperbolic paraboloid; (*c*) the elliptic cone.

2. The line perpendicular to a tangent plane at a point of contact is called the *normal line.* Write the equations of the tangent planes and normal lines to the following quadric surfaces at the points indicated:

(*a*) $x^2/9+y^2/4-z^2/16=1$, at $(3,\,-1,\,2)$;
(*b*) $x^2+2\,y^2+z^2=10$, at $(2,\,1,\,-2)$;
(*c*) $x^2+2\,y^2-2\,z^2=0$, at $(4,\,1,\,3)$; (*d*) $x^2-3\,y^2-z=0$, at the origin.

3. Show that the cylinder whose axis has the direction cosines $l,\ m,\ n$ and which is tangent to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, is

$$\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0.$$

4. Show that the plane $lx+my+nz=\sqrt{l^2a^2+m^2b^2+n^2c^2}$ is tangent to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

5. Show that the locus of the intersection of three mutually perpendicular tangent planes to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, is the sphere (called *director sphere*) $x^2+y^2+z^2=a^2+b^2+c^2$.

6. Show that the elliptic cone is a ruled surface.

7. Show that any two linear equations which contain a parameter represent the generating line of a ruled surface. What surfaces are generated by the following lines?

$$(a)\ x - y + kz = 0,\ x + y - z/k = 0;\ (b)\ 3x - 4y = k,\ (3x + 4y)k = 1;$$
$$(c)\ x - y + 3kz = 3k,\ k(x + y) - z = 3.$$

8. Show that every generating line of the hyperbolic paraboloid $x^2/a^2 - y^2/b^2 = 2cz$ is parallel to one of the planes $x^2/a^2 - y^2/b^2 = 0$,

**284. Surfaces in General.** When it is required to determine the shape of a surface from its cartesian equation

$$F(x, y, z) = 0,$$

the most effective methods, apart from the calculus, are the transformation of coordinates and the taking of cross-sections, generally (though not necessarily always) at right angles to the axes of coordinates. Both these methods have been applied repeatedly to the quadric surfaces in the preceding articles.

**285. Cross-Sections.** The method of cross-sections is extensively used in the applications. The railroad engineer determines thus the shape of a railroad dam; the naval architect uses it in laying out his ship; even the biologist uses it in constructing enlarged models of small organs of plants or animals.

**286. Parallel Planes.** When the given equation contains only one of the variables $x$, $y$, $z$, it represents of course a set of *parallel planes* (real or imaginary), at right angles to one of the axes. Thus any equation of the form

$$F(x) = 0$$

represents planes at right angles to $Ox$, of which as many are real as the equation has real roots.

**287. Cylinders.** When the given equation contains only two variables it represents a *cylinder* at right angles to one of the coordinate planes. Thus any equation of the form

$$F(x, y) = 0$$

represents a cylinder passing through the curve $F(x, y) = 0$ in the plane $Oxy$, with generators parallel to $Oz$. If, in particular, $F(x, y)$ is homogeneous in $x$ and $y$, *i.e.* if all terms are of the same degree, the cylinder breaks up into planes.

**288. Cones.** When the given equation $F(x, y, z) = 0$ is homogeneous in $x$, $y$, and $z$, *i.e.* if all terms are of the same degree, the equation represents a general *cone*, with vertex at the origin. For in this case, if $(x, y, z)$ is a point of the surface, so is the point $(kx, ky, kz)$, where $k$ is any constant; in other words, if $P$ is a point of the surface, then every point of the line $OP$ belongs to the surface; the surface can therefore be generated by the motion of a line passing through the origin

**289. Functions of Two Variables.** Just as plane curves are used to represent functions of a single variable, so surfaces can be used to represent ***functions of two variables.*** Thus to obtain an intuitive picture of a given function $f(x, y)$ we may construct a *model* of the surface

$$z = f(x, y),$$

such as the relief map of a mountainous country. The ordinate $z$ of the surface represents the function.

**290. Contour Lines.** To obtain some idea of such a surface by means of a *plane* drawing the method of ***contour lines*** or ***level lines*** can be used. This is done, *e.g.*, in topographical maps. The method consists in taking horizontal cross-sections at equal intervals and projecting these cross-sections on the horizontal plane. Where the level lines crowd together the surface is steep; where they are relatively far apart the surface is flat.

### EXERCISES

**1.** What surfaces are represented by the following equations?

(*a*) $Ax + By + C = 0$.
(*b*) $x \cos\beta + y \sin\beta = p$.
(*c*) $y^2 + z^2 = a^2$.
(*d*) $z^2 - x^2 = a^2$.
(*e*) $zx = a^2$.
(*f*) $z^2 = 4\,ay$.
(*g*) $x^3 - 3\,x^2 - x + 3 = 0$.
(*h*) $xyz = 0$.
(*i*) $y = x^2 - x - 6$.
(*j*) $yz^2 - 9\,y = 0$.
(*k*) $x^2 + 2\,y^2 = 0$.
(*l*) $x^2 = yz$.
(*m*) $x^2 - y^2 = z^2$.
(*n*) $y^2 + 2\,z^2 + 4\,zx = 0$.
(*o*) $(x - 1)(y - 2)(z - 3) = 0$.
(*p*) $x^3 + y^3 - 3\,xyz = 0$.

**2.** Determine the nature of the following surfaces by sketching the contour lines:

(*a*) $z = x + y$.
(*b*) $z = xy$.
(*c*) $z = y/x$.
(*d*) $z = x^2 + y$.
(*e*) $z = x^2 - y^2 + 4$.
(*f*) $z = x^2$.
(*g*) $z = x^2 + y^2 - 4\,x$.
(*h*) $z = xy - x$.
(*i*) $z = 2^x$.
(*j*) $y = z^2 - 4\,x$.
(*k*) $y = 3\,z^2 + x^2$.
(*l*) $z = 3\,x + y^2$.

**3.** The Cassinian ovals (§ 178) are contour lines of what surface?

**4.** What can be said about the nature of the contour lines of a surface $z = f(x)$? Discuss in particular: (*a*) $z = x^2 - 9$; (*b*) $z = x^3 - 8$; (*c*) $y = z^2 + 2\,z$.

**291. Rotation of Coordinate Trihedral.** To transform the equation of a surface from one coordinate trihedral $Oxyz$ to another $Ox'y'z'$, with the same origin $O$, we must find expressions for the *old* coordinates $x$, $y$, $z$ of any point $P$ in terms of the *new* coordinates $x'$, $y'$, $z'$. We here confine ourselves to the case when each trihedral is trirectangular; this is the case of *orthogonal transformation*, or *orthogonal substitution*.

Let $l_1$, $m_1$, $n_1$ be the direction cosines of the new axis $Ox'$ with respect to the old axes $Ox$, $Oy$, $Oz$ (Fig. 135); similarly $l_2$, $m_2$, $n_2$ those of $Oy'$, and $l_3$, $m_3$, $n_3$ those of $Oz'$. This is indicated by the scheme

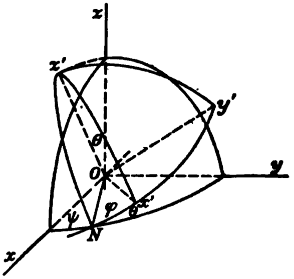

FIG. 135

$$\begin{array}{c|ccc} & x' & y' & z' \\ \hline x & l_1 & l_2 & l_3 \\ y & m_1 & m_2 & m_3 \\ z & n_1 & n_2 & n_3 \end{array}$$

which shows at the same time that then the direction cosines of the old axis $Ox$ with respect to the new axes $Ox'$, $Oy'$, $Oz'$ are $l_1$, $l_2$, $l_3$, etc.

**292.** The nine direction cosines $l_1, l_2, \cdots n_3$ are sufficient to determine the position of the new trihedral $Ox'y'z'$ with respect to the old. But these nine quantities cannot be selected arbitrarily; they are connected by six independent relations which can be written in either of the equivalent forms

$$(1) \qquad \begin{aligned} &l_1^2 + m_1^2 + n_1^2 = 1, &\quad l_2l_3 + m_2m_3 + n_2n_3 = 0,\\ &l_2^2 + m_2^2 + n_2^2 = 1, &\quad l_3l_1 + m_3m_1 + n_3n_1 = 0,\\ &l_3^2 + m_3^2 + n_3^2 = 1, &\quad l_1l_2 + m_1m_2 + n_1n_2 = 0, \end{aligned}$$

or

$$(1') \qquad \begin{aligned} &l_1^2 + l_2^2 + l_3^2 = 1, &\quad m_1n_1 + m_2n_2 + m_3n_3 = 0,\\ &m_1^2 + m_2^2 + m_3^2 = 1, &\quad n_1l_1 + n_2l_2 + n_3l_3 = 0,\\ &n_1^2 + n_2^2 + n_3^2 = 1, &\quad l_1m_1 + l_2m_2 + l_3m_3 = 0. \end{aligned}$$

The meaning of these equations follows from §§ 202 and 205. Thus the first of the equations (1) expresses the fact that $l_1$, $m_1$, $n_1$ are the direction cosines of a line, viz. $Ox'$; the last of the equations (1′) expresses the perpendicularity of the axes $Ox$ and $Oy$; and so on.

**293.** If $x$, $y$, $z$ are the old, $x'$, $y'$, $z'$ the new coordinates of one and the same point, we find by observing that the projection on $Ox$ of the radius vector of $P$ is equal to the sum of the projections on $Ox$ of its components $x'$, $y'$, $z'$ (§ 199), and similarly for the projections on $Oy$ and $Oz$:

$$(2) \qquad \begin{aligned} x &= l_1x' + l_2y' + l_3z',\\ y &= m_1x' + m_2y' + m_3z',\\ z &= n_1x' + n_2y' + n_3z'. \end{aligned}$$

Indeed, these relations can be directly read off from the scheme of direction cosines in § 291.

Likewise, projecting on $Ox'$, $Oy'$, $Oz'$, we find

$$(2') \qquad \begin{aligned} x' &= l_1x + m_1y + n_1z,\\ y' &= l_2x + m_2y + n_2z,\\ z' &= l_3x + m_3y + n_3z. \end{aligned}$$

As the equations (2), by means of which we can transform the equation of any surface from one rectangular system of coordinates to any other with the same origin, give $x, y, z$ as *linear* functions of $x', y', z'$, it follows that *such a transformation cannot change the degree of the equation of the surface.*

**294.** The equation (2′) must of course result also by solving the equations (2) for $x', y', z'$, and *vice versa*. Putting

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = D,$$

solving (2) for $x', y', z'$, and comparing the coefficients of $x, y, z$ with those in (2′) we find the following relations:

$$Dl_1 = m_2n_3 - m_3n_2, \quad Dm_1 = n_2l_3 - n_3l_2, \quad Dn_1 = l_2m_3 - l_3m_2, \quad \text{etc.}$$

Squaring and adding the first three equations and applying the relations (1) we find: $D^2 = 1$.

By § 226, $D$ can be interpreted as six times the volume of the tetrahedron whose vertices are the origin and the points $x', y', z'$ in Fig. 135, *i.e.* the intersections of the new axes with the unit sphere about the origin. The determinant gives this volume with the sign + or − according as the trihedral $Ox'y'z'$ is superposable or not (in direction and sense) to the trihedral $Oxyz$ (see § 295). It follows that $D = \pm 1$ and

$$\begin{aligned} l_1 &= \pm (m_2n_3 - m_3n_2), & m_1 &= \pm (n_2l_3 - n_3l_2), & n_1 &= \pm (l_2m_3 - l_3m_2), \\ l_2 &= \pm (m_3n_1 - m_1n_3), & m_2 &= \pm (n_3l_1 - n_1l_3), & n_2 &= \pm (l_3m_1 - l_1m_3), \\ l_3 &= \pm (m_1n_2 - m_2n_1), & m_3 &= \pm (n_1l_2 - n_2l_1), & n_3 &= \pm (l_1m_2 - l_2m_1), \end{aligned}$$

the upper or lower signs to be used according as the trihedrals are superposable or not.

**295.** A rectangular trihedral $Oxyz$ is called ***right-handed*** if the rotation that turns $Oy$ through 90° into $Oz$ appears *counterclockwise* as seen from $Ox$; otherwise it is called ***left-handed***. In the present work right-handed sets of axes have been used throughout.

Two right-handed as well as two left-handed rectangular trihedrals are superposable; a right-handed and a left-handed trihedral are not superposable. The difference is of the same kind as that between the gloves of the right and left hand.

Two non-superposable rectangular trihedrals become superposable upon reversing one (or all three) of the axes of either one.

**296.** The fact that the nine direction cosines are connected by six relations (§ 292) suggests that it must be possible to determine the position of the new trihedral with respect to the old by only three angles. As such we may take, in the case of superposable trihedrals, the angles $\theta$, $\phi$, $\psi$, marked in Fig. 135, which are known as ***Euler's angles.***

The figure shows the intersections of the two trihedrals with a sphere of radius 1 described about the origin as center. If $ON$ is the intersection of the planes $Oxy$ and $Ox'y'$, Euler's angles are defined as

$$\theta = zOz', \quad \phi = NOx', \quad \psi = xON.$$

The line $ON$ is called the *line of nodes*, or the ***nodal line.***

Imagine the new trihedral $Ox'y'z'$ initially coincident with the old trihedral $Oxyz$, in direction and sense. Now turn the new trihedral about $Oz$ in the positive (counterclockwise) sense until $Ox'$ coincides with the assumed positive sense of the nodal line $ON$; the amount of this rotation gives the angle $\psi$. Next turn the new trihedral about $ON$ in the positive sense until the plane $Ox'y'$ assumes its final position; this gives the angle $\theta$ as the angle between the planes $Oxy$ and $Ox'y'$, or the angle $zOz'$ between their normals. Finally a rotation of the new trihedral about the axis $Oz'$, which has reached its final position, in the positive sense until $Ox'$ assumes its final position, determines the angle $\phi$.

**297.** The relations between the nine direction cosines and the three angles of Euler are readily found from Fig. 135 by applying the fundamental formula of spherical trigonometry $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$ successively to the spherical triangles

$$\begin{array}{ccc} xNx', & xNy', & xNz', \\ yNx', & yNy', & yNz', \\ zNx', & zNy', & zNz'. \end{array}$$

We find in this way:

$$\begin{aligned} l_1 &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \\ m_1 &= \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta, \\ n_1 &= \sin\phi\sin\theta, \end{aligned}$$

$$\begin{aligned} l_2 &= -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta, & l_3 &= \sin\psi\sin\theta, \\ m_2 &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, & m_3 &= -\cos\psi\sin\theta, \\ n_2 &= \cos\phi\sin\theta, & n_3 &= \cos\theta. \end{aligned}$$

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **10** | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| **15** | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| **20** | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| **25** | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 4 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| **30** | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 2 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| **35** | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| **40** | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **45** | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **50** | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 2 | 4 | 5 | 6 | 7 | 8 | 9 |

**The proportional parts are stated in full for every tenth at the right-hand side. logarithm of any number of four significant figures can be read directly by add-**

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **55** | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **60** | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **65** | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **75** | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **85** | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **95** | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

ing the proportional part corresponding to the fourth figure to the tabular number corresponding to the first three figures. There may be an error of 1 in the last place.

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| Radians | Degrees | Sine | | Tangent | | Cotangent | | Cosine | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .0000 | **0° 00′** | .0000 | —— | .0000 | —— | —— | —— | 1.0000 | .0000 | **90° 00′** | 1.5708 |
| .0029 | 10 | .0029 | .4637 | .0029 | .4637 | 343.77 | .5363 | 1.0000 | .0000 | 50 | 1.5679 |
| .0058 | 20 | .0058 | .7648 | .0058 | .7648 | 171.89 | .2352 | 1.0000 | .0000 | 40 | 1.5650 |
| .0087 | 30 | .0087 | .9408 | .0087 | .9409 | 114.59 | .0591 | 1.0000 | .0000 | 30 | 1.5621 |
| .0116 | 40 | .0116 | .0658 | .0116 | .0658 | 85.940 | .9342 | .9999 | .0000 | 20 | 1.5592 |
| .0145 | 50 | .0145 | .1627 | .0145 | .1627 | 68.750 | .8373 | .9999 | .0000 | 10 | 1.5563 |
| .0175 | **1° 00′** | .0175 | .2419 | .0175 | .2419 | 57.290 | .7581 | .9998 | .9999 | **89° 00′** | 1.5533 |
| .0204 | 10 | .0204 | .3088 | .0204 | .3089 | 49.104 | .6911 | .9998 | .9999 | 50 | 1.5504 |
| .0233 | 20 | .0233 | .3668 | .0233 | .3669 | 42.964 | .6331 | .9997 | .9999 | 40 | 1.5475 |
| .0262 | 30 | .0262 | .4179 | .0262 | .4181 | 38.188 | .5819 | .9997 | .9999 | 30 | 1.5446 |
| .0291 | 40 | .0291 | .4637 | .0291 | .4638 | 34.368 | .5362 | .9996 | .9998 | 20 | 1.5417 |
| .0320 | 50 | .0320 | .5050 | .0320 | .5053 | 31.242 | .4947 | .9995 | .9998 | 10 | 1.5388 |
| .0349 | **2° 00′** | .0349 | .5428 | .0349 | .5431 | 28.636 | .4569 | .9994 | .9997 | **88° 00′** | 1.5359 |
| .0378 | 10 | .0378 | .5776 | .0378 | .5779 | 26.432 | .4221 | .9993 | .9997 | 50 | 1.5330 |
| .0407 | 20 | .0407 | .6097 | .0407 | .6101 | 24.542 | .3899 | .9992 | .9996 | 40 | 1.5301 |
| .0436 | 30 | .0436 | .6397 | .0437 | .6401 | 22.904 | .3599 | .9990 | .9996 | 30 | 1.5272 |
| .0465 | 40 | .0465 | .6677 | .0466 | .6682 | 21.470 | .3318 | .9989 | .9995 | 20 | 1.5243 |
| .0495 | 50 | .0494 | .6940 | .0495 | .6945 | 20.206 | .3055 | .9988 | .9995 | 10 | 1.5213 |
| .0524 | **3° 00′** | .0523 | .7188 | .0524 | .7194 | 19.081 | .2806 | .9986 | .9994 | **87° 00′** | 1.5184 |
| .0553 | 10 | .0552 | .7423 | .0553 | .7429 | 18.075 | .2571 | .9985 | .9993 | 50 | 1.5155 |
| .0582 | 20 | .0581 | .7645 | .0582 | .7652 | 17.169 | .2348 | .9983 | .9993 | 40 | 1.5126 |
| .0611 | 30 | .0610 | .7857 | .0612 | .7865 | 16.350 | .2135 | .9981 | .9992 | 30 | 1.5097 |
| .0640 | 40 | .0640 | .8059 | .0641 | .8067 | 15.605 | .1933 | .9980 | .9991 | 20 | 1.5068 |
| .0669 | 50 | .0669 | .8251 | .0670 | .8261 | 14.924 | .1739 | .9978 | .9990 | 10 | 1.5039 |
| .0698 | **4° 00′** | .0698 | .8436 | .0699 | .8446 | 14.301 | .1554 | .9976 | .9989 | **86° 00′** | 1.5010 |
| .0727 | 10 | .0727 | .8613 | .0729 | .8624 | 13.727 | .1376 | .9974 | .9989 | 50 | 1.4981 |
| .0756 | 20 | .0756 | .8783 | .0758 | .8795 | 13.197 | .1205 | .9971 | .9988 | 40 | 1.4952 |
| .0785 | 30 | .0785 | .8946 | .0787 | .8960 | 12.706 | .1040 | .9969 | .9987 | 30 | 1.4923 |
| .0814 | 40 | .0814 | .9104 | .0816 | .9118 | 12.251 | .0882 | .9967 | .9986 | 20 | 1.4893 |
| .0844 | 50 | .0843 | .9256 | .0846 | .9272 | 11.826 | .0728 | .9964 | .9985 | 10 | 1.4864 |
| .0873 | **5° 00′** | .0872 | .9403 | .0875 | .9420 | 11.430 | .0580 | .9962 | .9983 | **85° 00′** | 1.4835 |
| .0902 | 10 | .0901 | .9545 | .0904 | .9563 | 11.059 | .0437 | .9959 | .9982 | 50 | 1.4806 |
| .0931 | 20 | .0929 | .9682 | .0934 | .9701 | 10.712 | .0299 | .9957 | .9981 | 40 | 1.4777 |
| .0960 | 30 | .0958 | .9816 | .0963 | .9836 | 10.385 | .0164 | .9954 | .9980 | 30 | 1.4748 |
| .0989 | 40 | .0987 | .9945 | .0992 | .9966 | 10.078 | .0034 | .9951 | .9979 | 20 | 1.4719 |
| .1018 | 50 | .1016 | .0070 | .1022 | .0093 | 9.7882 | .9907 | .9948 | .9977 | 10 | 1.4690 |
| .1047 | **6° 00′** | .1045 | .0192 | .1051 | .0216 | 9.5144 | .9784 | .9945 | .9976 | **84° 00′** | 1.4661 |
| .1076 | 10 | .1074 | .0311 | .1080 | .0336 | 9.2553 | .9664 | .9942 | .9975 | 50 | 1.4632 |
| .1105 | 20 | .1103 | .0426 | .1110 | .0453 | 9.0098 | .9547 | .9939 | .9973 | 40 | 1.4603 |
| .1134 | 30 | .1132 | .0539 | .1139 | .0567 | 8.7769 | .9433 | .9936 | .9972 | 30 | 1.4573 |
| .1164 | 40 | .1161 | .0648 | .1169 | .0678 | 8.5555 | .9322 | .9932 | .9971 | 20 | 1.4544 |
| .1193 | 50 | .1190 | .0755 | .1198 | .0786 | 8.3450 | .9214 | .9929 | .9969 | 10 | 1.4515 |
| .1222 | **7° 00′** | .1219 | .0859 | .1228 | .0891 | 8.1443 | .9109 | .9925 | .9968 | **83° 00′** | 1.4486 |
| .1251 | 10 | .1248 | .0961 | .1257 | .0995 | 7.9530 | .9005 | .9922 | .9966 | 50 | 1.4457 |
| .1280 | 20 | .1276 | .1060 | .1287 | .1096 | 7.7704 | .8904 | .9918 | .9964 | 40 | 1.4428 |
| .1309 | 30 | .1305 | .1157 | .1317 | .1194 | 7.5958 | .8806 | .9914 | .9963 | 30 | 1.4399 |
| .1338 | 40 | .1334 | .1252 | .1346 | .1291 | 7.4287 | .8709 | .9911 | .9961 | 20 | 1.4370 |
| .1367 | 50 | .1363 | .1345 | .1376 | .1385 | 7.2687 | .8615 | .9907 | .9959 | 10 | 1.4341 |
| .1396 | **8° 00′** | .1392 | .1436 | .1405 | .1478 | 7.1154 | .8522 | .9903 | .9958 | **82° 00′** | 1.4312 |
| .1425 | 10 | .1421 | .1525 | .1435 | .1569 | 6.9682 | .8431 | .9899 | .9956 | 50 | 1.4283 |
| .1454 | 20 | .1449 | .1612 | .1465 | .1658 | 6.8269 | .8342 | .9894 | .9954 | 40 | 1.4254 |
| .1484 | 30 | .1478 | .1697 | .1495 | .1745 | 6.6912 | .8255 | .9890 | .9952 | 30 | 1.4224 |
| .1513 | 40 | .1507 | .1781 | .1524 | .1831 | 6.5606 | .8169 | .9886 | .9950 | 20 | 1.4195 |
| .1542 | 50 | .1536 | .1863 | .1554 | .1915 | 6.4348 | .8085 | .9881 | .9948 | 10 | 1.4166 |
| .1571 | **9° 00′** | .1564 | .1943 | .1584 | .1997 | 6.3138 | .8003 | .9877 | .9946 | **81° 00′** | 1.4137 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Degrees | Radians |
| | | Cosine | | Cotangent | | Tangent | | Sine | | | |

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .1571 | **9° 00′** | .1564 | .1943 | .1584 | .1997 | 6.3138 | .8003 | .9877 | .9946 | **81° 00′** | 1.4137 |
| .1600 | 10 | .1593 | .2022 | .1614 | .2078 | 6.1970 | .7922 | .9872 | .9944 | 50 | 1.4108 |
| .1629 | 20 | .1622 | .2100 | .1644 | .2158 | 6.0844 | .7842 | .9868 | .9942 | 40 | 1.4079 |
| .1658 | 30 | .1650 | .2176 | .1673 | .2236 | 5.9758 | .7764 | .9863 | .9940 | 30 | 1.4050 |
| .1687 | 40 | .1679 | .2251 | .1703 | .2313 | 5.8708 | .7687 | .9858 | .9938 | 20 | 1.4021 |
| .1716 | 50 | .1708 | .2324 | .1733 | .2389 | 5.7694 | .7611 | .9853 | .9936 | 10 | 1.3992 |
| .1745 | **10° 00′** | .1736 | .2397 | .1763 | .2463 | 5.6713 | .7537 | .9848 | .9934 | **80° 00′** | 1.3963 |
| .1774 | 10 | .1765 | .2468 | .1793 | .2536 | 5.5764 | .7464 | .9843 | .9931 | 50 | 1.3934 |
| .1804 | 20 | .1794 | .2538 | .1823 | .2609 | 5.4845 | .7391 | .9838 | .9929 | 40 | 1.3904 |
| .1833 | 30 | .1822 | .2606 | .1853 | .2680 | 5.3955 | .7320 | .9833 | .9927 | 30 | 1.3875 |
| .1862 | 40 | .1851 | .2674 | .1883 | .2750 | 5.3093 | .7250 | .9827 | .9924 | 20 | 1.3846 |
| .1891 | 50 | .1880 | .2740 | .1914 | .2819 | 5.2257 | .7181 | .9822 | .9922 | 10 | 1.3817 |
| .1920 | **11° 00′** | .1908 | .2806 | .1944 | .2887 | 5.1446 | .7113 | .9816 | .9919 | **79° 00′** | 1.3788 |
| .1949 | 10 | .1937 | .2870 | .1974 | .2953 | 5.0658 | .7047 | .9811 | .9917 | 50 | 1.3759 |
| .1978 | 20 | .1965 | .2934 | .2004 | .3020 | 4.9894 | .6980 | .9805 | .9914 | 40 | 1.3730 |
| .2007 | 30 | .1994 | .2997 | .2035 | .3085 | 4.9152 | .6915 | .9799 | .9912 | 30 | 1.3701 |
| .2036 | 40 | .2022 | .3058 | .2065 | .3149 | 4.8430 | .6851 | .9793 | .9909 | 20 | 1.3672 |
| .2065 | 50 | .2051 | .3119 | .2095 | .3212 | 4.7729 | .6788 | .9787 | .9907 | 10 | 1.3643 |
| .2094 | **12° 00′** | .2079 | .3179 | .2126 | .3275 | 4.7046 | .6725 | .9781 | .9904 | **78° 00′** | 1.3614 |
| .2123 | 10 | .2108 | .3238 | .2156 | .3336 | 4.6382 | .6664 | .9775 | .9901 | 50 | 1.3584 |
| .2153 | 20 | .2136 | .3296 | .2186 | .3397 | 4.5736 | .6603 | .9769 | .9899 | 40 | 1.3555 |
| .2182 | 30 | .2164 | .3353 | .2217 | .3458 | 4.5107 | .6542 | .9763 | .9896 | 30 | 1.3526 |
| .2211 | 40 | .2193 | .3410 | .2247 | .3517 | 4.4494 | .6483 | .9757 | .9893 | 20 | 1.3497 |
| .2240 | 50 | .2221 | .3466 | .2278 | .3576 | 4.3897 | .6424 | .9750 | .9890 | 10 | 1.3468 |
| .2269 | **13° 00′** | .2250 | .3521 | .2309 | .3634 | 4.3315 | .6366 | .9744 | .9887 | **77° 00′** | 1.3439 |
| .2298 | 10 | .2278 | .3575 | .2339 | .3691 | 4.2747 | .6309 | .9737 | .9884 | 50 | 1.3410 |
| .2327 | 20 | .2306 | .3629 | .2370 | .3748 | 4.2193 | .6252 | .9730 | .9881 | 40 | 1.3381 |
| .2356 | 30 | .2334 | .3682 | .2401 | .3804 | 4.1653 | .6196 | .9724 | .9878 | 30 | 1.3352 |
| .2385 | 40 | .2363 | .3734 | .2432 | .3859 | 4.1126 | .6141 | .9717 | .9875 | 20 | 1.3323 |
| .2414 | 50 | .2391 | .3786 | .2462 | .3914 | 4.0611 | .6086 | .9710 | .9872 | 10 | 1.3294 |
| .2443 | **14° 00′** | .2419 | .3837 | .2493 | .3968 | 4.0108 | .6032 | .9703 | .9869 | **76° 00′** | 1.3265 |
| .2473 | 10 | .2447 | .3887 | .2524 | .4021 | 3.9617 | .5979 | .9696 | .9866 | 50 | 1.3235 |
| .2502 | 20 | .2476 | .3937 | .2555 | .4074 | 3.9136 | .5926 | .9689 | .9863 | 40 | 1.3206 |
| .2531 | 30 | .2504 | .3986 | .2586 | .4127 | 3.8667 | .5873 | .9681 | .9859 | 30 | 1.3177 |
| .2560 | 40 | .2532 | .4035 | .2617 | .4178 | 3.8208 | .5822 | .9674 | .9856 | 20 | 1.3148 |
| .2589 | 50 | .2560 | .4083 | .2648 | .4230 | 3.7760 | .5770 | .9667 | .9853 | 10 | 1.3119 |
| .2618 | **15° 00′** | .2588 | .4130 | .2679 | .4281 | 3.7321 | .5719 | .9659 | .9849 | **75° 00′** | 1.3090 |
| .2647 | 10 | .2616 | .4177 | .2711 | .4331 | 3.6891 | .5669 | .9652 | .9846 | 50 | 1.3061 |
| .2676 | 20 | .2644 | .4223 | .2742 | .4381 | 3.6470 | .5619 | .9644 | .9843 | 40 | 1.3032 |
| .2705 | 30 | .2672 | .4269 | .2773 | .4430 | 3.6059 | .5570 | .9636 | .9839 | 30 | 1.3003 |
| .2734 | 40 | .2700 | .4314 | .2805 | .4479 | 3.5656 | .5521 | .9628 | .9836 | 20 | 1.2974 |
| .2763 | 50 | .2728 | .4359 | .2836 | .4527 | 3.5261 | .5473 | .9621 | .9832 | 10 | 1.2945 |
| .2793 | **16° 00′** | .2756 | .4403 | .2867 | .4575 | 3.4874 | .5425 | .9613 | .9828 | **74° 00′** | 1.2915 |
| .2822 | 10 | .2784 | .4447 | .2899 | .4622 | 3.4495 | .5378 | .9605 | .9825 | 50 | 1.2886 |
| .2851 | 20 | .2812 | .4491 | .2931 | .4669 | 3.4124 | .5331 | .9596 | .9821 | 40 | 1.2857 |
| .2880 | 30 | .2840 | .4533 | .2962 | .4716 | 3.3759 | .5284 | .9588 | .9817 | 30 | 1.2828 |
| .2909 | 40 | .2868 | .4576 | .2994 | .4762 | 3.3402 | .5238 | .9580 | .9814 | 20 | 1.2799 |
| .2938 | 50 | .2896 | .4618 | .3026 | .4808 | 3.3052 | .5192 | .9572 | .9810 | 10 | 1.2770 |
| .2967 | **17° 00′** | .2924 | .4659 | .3057 | .4853 | 3.2709 | .5147 | .9563 | .9806 | **73° 00′** | 1.2741 |
| .2996 | 10 | .2952 | .4700 | .3089 | .4898 | 3.2371 | .5102 | .9555 | .9802 | 50 | 1.2712 |
| .3025 | 20 | .2979 | .4741 | .3121 | .4943 | 3.2041 | .5057 | .9546 | .9798 | 40 | 1.2683 |
| .3054 | 30 | .3007 | .4781 | .3153 | .4987 | 3.1716 | .5013 | .9537 | .9794 | 30 | 1.2654 |
| .3083 | 40 | .3035 | .4821 | .3185 | .5031 | 3.1397 | .4969 | .9528 | .9790 | 20 | 1.2625 |
| .3113 | 50 | .3062 | .4861 | .3217 | .5075 | 3.1084 | .4925 | .9520 | .9786 | 10 | 1.2595 |
| .3142 | **18° 00′** | .3090 | .4900 | .3249 | .5118 | 3.0777 | .4882 | .9511 | .9782 | **72° 00′** | 1.2566 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIA |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| Radians | Degrees | Sine | | Tangent | | Cotangent | | Cosine | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .3142 | **18° 00′** | .3090 | .4900 | .3249 | .5118 | 3.0777 | .4882 | .9511 | .9782 | **72° 00′** | 1.2566 |
| .3171 | 10 | .3118 | .4939 | .3281 | .5161 | 3.0475 | .4839 | .9502 | .9778 | 50 | 1.2537 |
| .3200 | 20 | .3145 | .4977 | .3314 | .5203 | 3.0178 | .4797 | .9492 | .9774 | 40 | 1.2508 |
| .3229 | 30 | .3173 | .5015 | .3346 | .5245 | 2.9887 | .4755 | .9483 | .9770 | 30 | 1.2479 |
| .3258 | 40 | .3201 | .5052 | .3378 | .5287 | 2.9600 | .4713 | .9474 | .9765 | 20 | 1.2450 |
| .3287 | 50 | .3228 | .5090 | .3411 | .5329 | 2.9319 | .4671 | .9465 | .9761 | 10 | 1.2421 |
| .3316 | **19° 00′** | .3256 | .5126 | .3443 | .5370 | 2.9042 | .4630 | .9455 | .9757 | **71° 00′** | 1.2392 |
| .3345 | 10 | .3283 | .5163 | .3476 | .5411 | 2.8770 | .4589 | .9446 | .9752 | 50 | 1.2363 |
| .3374 | 20 | .3311 | .5199 | .3508 | .5451 | 2.8502 | .4549 | .9436 | .9748 | 40 | 1.2334 |
| .3403 | 30 | .3338 | .5235 | .3541 | .5491 | 2.8239 | .4509 | .9426 | .9743 | 30 | 1.2305 |
| .3432 | 40 | .3365 | .5270 | .3574 | .5531 | 2.7980 | .4469 | .9417 | .9739 | 20 | 1.2275 |
| .3462 | 50 | .3393 | .5306 | .3607 | .5571 | 2.7725 | .4429 | .9407 | .9734 | 10 | 1.2246 |
| .3491 | **20° 00′** | .3420 | .5341 | .3640 | .5611 | 2.7475 | .4389 | .9397 | .9730 | **70° 00′** | 1.2217 |
| .3520 | 10 | .3448 | .5375 | .3673 | .5650 | 2.7228 | .4350 | .9387 | .9725 | 50 | 1.2188 |
| .3549 | 20 | .3475 | .5409 | .3706 | .5689 | 2.6985 | .4311 | .9377 | .9721 | 40 | 1.2159 |
| .3578 | 30 | .3502 | .5443 | .3739 | .5727 | 2.6746 | .4273 | .9367 | .9716 | 30 | 1.2130 |
| .3607 | 40 | .3529 | .5477 | .3772 | .5766 | 2.6511 | .4234 | .9356 | .9711 | 20 | 1.2101 |
| .3636 | 50 | .3557 | .5510 | .3805 | .5804 | 2.6279 | .4196 | .9346 | .9706 | 10 | 1.2072 |
| .3665 | **21° 00′** | .3584 | .5543 | .3839 | .5842 | 2.6051 | .4158 | .9336 | .9702 | **69° 00′** | 1.2043 |
| .3694 | 10 | .3611 | .5576 | .3872 | .5879 | 2.5826 | .4121 | .9325 | .9697 | 50 | 1.2014 |
| .3723 | 20 | .3638 | .5609 | .3906 | .5917 | 2.5605 | .4083 | .9315 | .9692 | 40 | 1.1985 |
| .3752 | 30 | .3665 | .5641 | .3939 | .5954 | 2.5386 | .4046 | .9304 | .9687 | 30 | 1.1956 |
| .3782 | 40 | .3692 | .5673 | .3973 | .5991 | 2.5172 | .4009 | .9293 | .9682 | 20 | 1.1926 |
| .3811 | 50 | .3719 | .5704 | .4006 | .6028 | 2.4960 | .3972 | .9283 | .9677 | 10 | 1.1897 |
| .3840 | **22° 00′** | .3746 | .5736 | .4040 | .6064 | 2.4751 | .3936 | .9272 | .9672 | **68° 00′** | 1.1868 |
| .3869 | 10 | .3773 | .5767 | .4074 | .6100 | 2.4545 | .3900 | .9261 | .9667 | 50 | 1.1839 |
| .3898 | 20 | .3800 | .5798 | .4108 | .6136 | 2.4342 | .3864 | .9250 | .9661 | 40 | 1.1810 |
| .3927 | 30 | .3827 | .5828 | .4142 | .6172 | 2.4142 | .3828 | .9239 | .9656 | 30 | 1.1781 |
| .3956 | 40 | .3854 | .5859 | .4176 | .6208 | 2.3945 | .3792 | .9228 | .9651 | 20 | 1.1752 |
| .3985 | 50 | .3881 | .5889 | .4210 | .6243 | 2.3750 | .3757 | .9216 | .9646 | 10 | 1.1723 |
| .4014 | **23° 00′** | .3907 | .5919 | .4245 | .6279 | 2.3559 | .3721 | .9205 | .9640 | **67° 00′** | 1.1694 |
| .4043 | 10 | .3934 | .5948 | .4279 | .6314 | 2.3369 | .3686 | .9194 | .9635 | 50 | 1.1665 |
| .4072 | 20 | .3961 | .5978 | .4314 | .6348 | 2.3183 | .3652 | .9182 | .9629 | 40 | 1.1636 |
| .4102 | 30 | .3987 | .6007 | .4348 | .6383 | 2.2998 | .3617 | .9171 | .9624 | 30 | 1.1606 |
| .4131 | 40 | .4014 | .6036 | .4383 | .6417 | 2.2817 | .3583 | .9159 | .9618 | 20 | 1.1577 |
| .4160 | 50 | .4041 | .6065 | .4417 | .6452 | 2.2637 | .3548 | .9147 | .9613 | 10 | 1.1548 |
| .4189 | **24° 00′** | .4067 | .6093 | .4452 | .6486 | 2.2460 | .3514 | .9135 | .9607 | **66° 00′** | 1.1519 |
| .4218 | 10 | .4094 | .6121 | .4487 | .6520 | 2.2286 | .3480 | .9124 | .9602 | 50 | 1.1490 |
| .4247 | 20 | .4120 | .6149 | .4522 | .6553 | 2.2113 | .3447 | .9112 | .9596 | 40 | 1.1461 |
| .4276 | 30 | .4147 | .6177 | .4557 | .6587 | 2.1943 | .3413 | .9100 | .9590 | 30 | 1.1432 |
| .4305 | 40 | .4173 | .6205 | .4592 | .6620 | 2.1775 | .3380 | .9088 | .9584 | 20 | 1.1403 |
| .4334 | 50 | .4200 | .6232 | .4628 | .6654 | 2.1609 | .3346 | .9075 | .9579 | 10 | 1.1374 |
| .4363 | **25° 00′** | .4226 | .6259 | .4663 | .6687 | 2.1445 | .3313 | .9063 | .9573 | **65° 00′** | 1.1345 |
| .4392 | 10 | .4253 | .6286 | .4699 | .6720 | 2.1283 | .3280 | .9051 | .9567 | 50 | 1.1316 |
| .4422 | 20 | .4279 | .6313 | .4734 | .6752 | 2.1123 | .3248 | .9038 | .9561 | 40 | 1.1286 |
| .4451 | 30 | .4305 | .6340 | .4770 | .6785 | 2.0965 | .3215 | .9026 | .9555 | 30 | 1.1257 |
| .4480 | 40 | .4331 | .6366 | .4806 | .6817 | 2.0809 | .3183 | .9013 | .9549 | 20 | 1.1228 |
| .4509 | 50 | .4358 | .6392 | .4841 | .6850 | 2.0655 | .3150 | .9001 | .9543 | 10 | 1.1199 |
| .4538 | **26° 00′** | .4384 | .6418 | .4877 | .6882 | 2.0503 | .3118 | .8988 | .9537 | **64° 00′** | 1.1170 |
| .4567 | 10 | .4410 | .6444 | .4913 | .6914 | 2.0353 | .3086 | .8975 | .9530 | 50 | 1.1141 |
| .4596 | 20 | .4436 | .6470 | .4950 | .6946 | 2.0204 | .3054 | .8962 | .9524 | 40 | 1.1112 |
| .4625 | 30 | .4462 | .6495 | .4986 | .6977 | 2.0057 | .3023 | .8949 | .9518 | 30 | 1.1083 |
| .4654 | 40 | .4488 | .6521 | .5022 | .7009 | 1.9912 | .2991 | .8936 | .9512 | 20 | 1.1054 |
| .4683 | 50 | .4514 | .6546 | .5059 | .7040 | 1.9768 | .2960 | .8923 | .9505 | 10 | 1.1025 |
| .4712 | **27° 00′** | .4540 | .6570 | .5095 | .7072 | 1.9626 | .2928 | .8910 | .9499 | **63° 00′** | 1.0996 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Degrees | Radians |
| | | Cosine | | Cotangent | | Tangent | | Sine | | | |

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .4712 | **27° 00′** | .4540 | .6570 | .5095 | .7072 | 1.9626 | .2928 | .8910 | .9499 | **63° 00′** | 1.0996 |
| .4741 | 10 | .4566 | .6595 | .5132 | .7103 | 1.9486 | .2897 | .8897 | .9492 | 50 | 1.0966 |
| .4771 | 20 | .4592 | .6620 | .5169 | .7134 | 1.9347 | .2866 | .8884 | .9486 | 40 | 1.0937 |
| .4800 | 30 | .4617 | .6644 | .5206 | .7165 | 1.9210 | .2835 | .8870 | .9479 | 30 | 1.0908 |
| .4829 | 40 | .4643 | .6668 | .5243 | .7196 | 1.9074 | .2804 | .8857 | .9473 | 20 | 1.0879 |
| .4858 | 50 | .4669 | .6692 | .5280 | .7226 | 1.8940 | .2774 | .8843 | .9466 | 10 | 1.0850 |
| .4887 | **28° 00′** | .4695 | .6716 | .5317 | .7257 | 1.8807 | .2743 | .8829 | .9459 | **62° 00′** | 1.0821 |
| .4916 | 10 | .4720 | .6740 | .5354 | .7287 | 1.8676 | .2713 | .8816 | .9453 | 50 | 1.0792 |
| .4945 | 20 | .4746 | .6763 | .5392 | .7317 | 1.8546 | .2683 | .8802 | .9446 | 40 | 1.0763 |
| .4974 | 30 | .4772 | .6787 | .5430 | .7348 | 1.8418 | .2652 | .8788 | .9439 | 30 | 1.0734 |
| .5003 | 40 | .4797 | .6810 | .5467 | .7378 | 1.8291 | .2622 | .8774 | .9432 | 20 | 1.0705 |
| .5032 | 50 | .4823 | .6833 | .5505 | .7408 | 1.8165 | .2592 | .8760 | .9425 | 10 | 1.0676 |
| .5061 | **29° 00′** | .4848 | .6856 | .5543 | .7438 | 1.8040 | .2562 | .8746 | .9418 | **61° 00′** | 1.0647 |
| .5091 | 10 | .4874 | .6878 | .5581 | .7467 | 1.7917 | .2533 | .8732 | .9411 | 50 | 1.0617 |
| .5120 | 20 | .4899 | .6901 | .5619 | .7497 | 1.7796 | .2503 | .8718 | .9404 | 40 | 1.0588 |
| .5149 | 30 | .4924 | .6923 | .5658 | .7526 | 1.7675 | .2474 | .8704 | .9397 | 30 | 1.0559 |
| .5178 | 40 | .4950 | .6946 | .5696 | .7556 | 1.7556 | .2444 | .8689 | .9390 | 20 | 1.0530 |
| .5207 | 50 | .4975 | .6968 | .5735 | .7585 | 1.7437 | .2415 | .8675 | .9383 | 10 | 1.0501 |
| .5236 | **30° 00′** | .5000 | .6990 | .5774 | .7614 | 1.7321 | .2386 | .8660 | .9375 | **60° 00′** | 1.0472 |
| .5265 | 10 | .5025 | .7012 | .5812 | .7644 | 1.7205 | .2356 | .8646 | .9368 | 50 | 1.0443 |
| .5294 | 20 | .5050 | .7033 | .5851 | .7673 | 1.7090 | .2327 | .8631 | .9361 | 40 | 1.0414 |
| .5323 | 30 | .5075 | .7055 | .5890 | .7701 | 1.6977 | .2299 | .8616 | .9353 | 30 | 1.0385 |
| .5352 | 40 | .5100 | .7076 | .5930 | .7730 | 1.6864 | .2270 | .8601 | .9346 | 20 | 1.0356 |
| .5381 | 50 | .5125 | .7097 | .5969 | .7759 | 1.6753 | .2241 | .8587 | .9338 | 10 | 1.0327 |
| .5411 | **31° 00′** | .5150 | .7118 | .6009 | .7788 | 1.6643 | .2212 | .8572 | .9331 | **59° 00′** | 1.0297 |
| .5440 | 10 | .5175 | .7139 | .6048 | .7816 | 1.6534 | .2184 | .8557 | .9323 | 50 | 1.0268 |
| .5469 | 20 | .5200 | .7160 | .6088 | .7845 | 1.6426 | .2155 | .8542 | .9315 | 40 | 1.0239 |
| .5498 | 30 | .5225 | .7181 | .6128 | .7873 | 1.6319 | .2127 | .8526 | .9308 | 30 | 1.0210 |
| .5527 | 40 | .5250 | .7201 | .6168 | .7902 | 1.6212 | .2098 | .8511 | .9300 | 20 | 1.0181 |
| .5556 | 50 | .5275 | .7222 | .6208 | .7930 | 1.6107 | .2070 | .8496 | .9292 | 10 | 1.0152 |
| .5585 | **32° 00′** | .5299 | .7242 | .6249 | .7958 | 1.6003 | .2042 | .8480 | .9284 | **58° 00′** | 1.0123 |
| .5614 | 10 | .5324 | .7262 | .6289 | .7986 | 1.5900 | .2014 | .8465 | .9276 | 50 | 1.0094 |
| .5643 | 20 | .5348 | .7282 | .6330 | .8014 | 1.5798 | .1986 | .8450 | .9268 | 40 | 1.0065 |
| .5672 | 30 | .5373 | .7302 | .6371 | .8042 | 1.5697 | .1958 | .8434 | .9260 | 30 | 1.0036 |
| .5701 | 40 | .5398 | .7322 | .6412 | .8070 | 1.5597 | .1930 | .8418 | .9252 | 20 | 1.0007 |
| .5730 | 50 | .5422 | .7342 | .6453 | .8097 | 1.5497 | .1903 | .8403 | .9244 | 10 | .9977 |
| .5760 | **33° 00′** | .5446 | .7361 | .6494 | .8125 | 1.5399 | .1875 | .8387 | .9236 | **57° 00′** | .9948 |
| .5789 | 10 | .5471 | .7380 | .6536 | .8153 | 1.5301 | .1847 | .8371 | .9228 | 50 | .9919 |
| .5818 | 20 | .5495 | .7400 | .6577 | .8180 | 1.5204 | .1820 | .8355 | .9219 | 40 | .9890 |
| .5847 | 30 | .5519 | .7419 | .6619 | .8208 | 1.5108 | .1792 | .8339 | .9211 | 30 | .9861 |
| .5876 | 40 | .5544 | .7438 | .6661 | .8235 | 1.5013 | .1765 | .8323 | .9203 | 20 | .9832 |
| .5905 | 50 | .5568 | .7457 | .6703 | .8263 | 1.4919 | .1737 | .8307 | .9194 | 10 | .9803 |
| .5934 | **34° 00′** | .5592 | .7476 | .6745 | .8290 | 1.4826 | .1710 | .8290 | .9186 | **56° 00′** | .9774 |
| .5963 | 10 | .5616 | .7494 | .6787 | .8317 | 1.4733 | .1683 | .8274 | .9177 | 50 | .9745 |
| .5992 | 20 | .5640 | .7513 | .6830 | .8344 | 1.4641 | .1656 | .8258 | .9169 | 40 | .9716 |
| .6021 | 30 | .5664 | .7531 | .6873 | .8371 | 1.4550 | .1629 | .8241 | .9160 | 30 | .9687 |
| .6050 | 40 | .5688 | .7550 | .6916 | .8398 | 1.4460 | .1602 | .8225 | .9151 | 20 | .9657 |
| .6080 | 50 | .5712 | .7568 | .6959 | .8425 | 1.4370 | .1575 | .8208 | .9142 | 10 | .9628 |
| .6109 | **35° 00′** | .5736 | .7586 | .7002 | .8452 | 1.4281 | .1548 | .8192 | .9134 | **55° 00′** | .9599 |
| .6138 | 10 | .5760 | .7604 | .7046 | .8479 | 1.4193 | .1521 | .8175 | .9125 | 50 | .9570 |
| .6167 | 20 | .5783 | .7622 | .7089 | .8506 | 1.4106 | .1494 | .8158 | .9116 | 40 | .9541 |
| .6196 | 30 | .5807 | .7640 | .7133 | .8533 | 1.4019 | .1467 | .8141 | .9107 | 30 | .9512 |
| .6225 | 40 | .5831 | .7657 | .7177 | .8559 | 1.3934 | .1441 | .8124 | .9098 | 20 | .9483 |
| .6254 | 50 | .5854 | .7675 | .7221 | .8586 | 1.3848 | .1414 | .8107 | .9089 | 10 | .9454 |
| .6283 | **36° 00′** | .5878 | .7692 | .7265 | .8613 | 1.3764 | .1387 | .8090 | .9080 | **54° 00′** | .9425 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIANS |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .6283 | **36° 00′** | .5878 | .7692 | .7265 | .8613 | 1.3764 | .1387 | .8090 | .9080 | **54° 00′** | .9425 |
| .6312 | 10 | .5901 | .7710 | .7310 | .8639 | 1.3680 | .1361 | .8073 | .9070 | 50 | .9396 |
| .6341 | 20 | .5925 | .7727 | .7355 | .8666 | 1.3597 | .1334 | .8056 | .9061 | 40 | .9367 |
| .6370 | 30 | .5948 | .7744 | .7400 | .8692 | 1.3514 | .1308 | .8039 | .9052 | 30 | .9338 |
| .6400 | 40 | .5972 | .7761 | .7445 | .8718 | 1.3432 | .1282 | .8021 | .9042 | 20 | .9308 |
| .6429 | 50 | .5995 | .7778 | .7490 | .8745 | 1.3351 | .1255 | .8004 | .9033 | 10 | .9279 |
| .6458 | **37° 00′** | .6018 | .7795 | .7536 | .8771 | 1.3270 | .1229 | .7986 | .9023 | **53° 00′** | .9250 |
| .6487 | 10 | .6041 | .7811 | .7581 | .8797 | 1.3190 | .1203 | .7969 | .9014 | 50 | .9221 |
| .6516 | 20 | .6065 | .7828 | .7627 | .8824 | 1.3111 | .1176 | .7951 | .9004 | 40 | .9192 |
| .6545 | 30 | .6088 | .7844 | .7673 | .8850 | 1.3032 | .1150 | .7934 | .8995 | 30 | .9163 |
| .6574 | 40 | .6111 | .7861 | .7720 | .8876 | 1.2954 | .1124 | .7916 | .8985 | 20 | .9134 |
| .6603 | 50 | .6134 | .7877 | .7766 | .8902 | 1.2876 | .1098 | .7898 | .8975 | 10 | .9105 |
| .6632 | **38° 00′** | .6157 | .7893 | .7813 | .8928 | 1.2799 | .1072 | .7880 | .8965 | **52° 00′** | .9076 |
| .6661 | 10 | .6180 | .7910 | .7860 | .8954 | 1.2723 | .1046 | .7862 | .8955 | 50 | .9047 |
| .6690 | 20 | .6202 | .7926 | .7907 | .8980 | 1.2647 | .1020 | .7844 | .8945 | 40 | .9018 |
| .6720 | 30 | .6225 | .7941 | .7954 | .9006 | 1.2572 | .0994 | .7826 | .8935 | 30 | .8988 |
| .6749 | 40 | .6248 | .7957 | .8002 | .9032 | 1.2497 | .0968 | .7808 | .8925 | 20 | .8959 |
| .6778 | 50 | .6271 | .7973 | .8050 | .9058 | 1.2423 | .0942 | .7790 | .8915 | 10 | .8930 |
| .6807 | **39° 00′** | .6293 | .7989 | .8098 | .9084 | 1.2349 | .0916 | .7771 | .8905 | **51° 00′** | .8901 |
| .6836 | 10 | .6316 | .8004 | .8146 | .9110 | 1.2276 | .0890 | .7753 | .8895 | 50 | .8872 |
| .6865 | 20 | .6338 | .8020 | .8195 | .9135 | 1.2203 | .0865 | .7735 | .8884 | 40 | .8843 |
| .6894 | 30 | .6361 | .8035 | .8243 | .9161 | 1.2131 | .0839 | .7716 | .8874 | 30 | .8814 |
| .6923 | 40 | .6383 | .8050 | .8292 | .9187 | 1.2059 | .0813 | .7698 | .8864 | 20 | .8785 |
| .6952 | 50 | .6406 | .8066 | .8342 | .9212 | 1.1988 | .0788 | .7679 | .8853 | 10 | .8756 |
| .6981 | **40° 00′** | .6428 | .8081 | .8391 | .9238 | 1.1918 | .0762 | .7660 | .8843 | **50° 00′** | .8727 |
| .7010 | 10 | .6450 | .8096 | .8441 | .9264 | 1.1847 | .0736 | .7642 | .8832 | 50 | .8698 |
| .7039 | 20 | .6472 | .8111 | .8491 | .9289 | 1.1778 | .0711 | .7623 | .8821 | 40 | .8668 |
| .7069 | 30 | .6494 | .8125 | .8541 | .9315 | 1.1708 | .0685 | .7604 | .8810 | 30 | .8639 |
| .7098 | 40 | .6517 | .8140 | .8591 | .9341 | 1.1640 | .0659 | .7585 | .8800 | 20 | .8610 |
| .7127 | 50 | .6539 | .8155 | .8642 | .9366 | 1.1571 | .0634 | .7566 | .8789 | 10 | .8581 |
| .7156 | **41° 00′** | .6561 | .8169 | .8693 | .9392 | 1.1504 | .0608 | .7547 | .8778 | **49° 00′** | .8552 |
| .7185 | 10 | .6583 | .8184 | .8744 | .9417 | 1.1436 | .0583 | .7528 | .8767 | 50 | .8523 |
| .7214 | 20 | .6604 | .8198 | .8796 | .9443 | 1.1369 | .0557 | .7509 | .8756 | 40 | .8494 |
| .7243 | 30 | .6626 | .8213 | .8847 | .9468 | 1.1303 | .0532 | .7490 | .8745 | 30 | .8465 |
| .7272 | 40 | .6648 | .8227 | .8899 | .9494 | 1.1237 | .0506 | .7470 | .8733 | 20 | .8436 |
| .7301 | 50 | .6670 | .8241 | .8952 | .9519 | 1.1171 | .0481 | .7451 | .8722 | 10 | .8407 |
| .7330 | **42° 00′** | .6691 | .8255 | .9004 | .9544 | 1.1106 | .0456 | .7431 | .8711 | **48° 00′** | .8378 |
| .7359 | 10 | .6713 | .8269 | .9057 | .9570 | 1.1041 | .0430 | .7412 | .8699 | 50 | .8348 |
| .7389 | 20 | .6734 | .8283 | .9110 | .9595 | 1.0977 | .0405 | .7392 | .8688 | 40 | .8319 |
| .7418 | 30 | .6756 | .8297 | .9163 | .9621 | 1.0913 | .0379 | .7373 | .8676 | 30 | .8290 |
| .7447 | 40 | .6777 | .8311 | .9217 | .9646 | 1.0850 | .0354 | .7353 | .8665 | 20 | .8261 |
| .7476 | 50 | .6799 | .8324 | .9271 | .9671 | 1.0786 | .0329 | .7333 | .8653 | 10 | .8232 |
| .7505 | **43° 00′** | .6820 | .8338 | .9325 | .9697 | 1.0724 | .0303 | .7314 | .8641 | **47° 00′** | .8203 |
| .7534 | 10 | .6841 | .8351 | .9380 | .9722 | 1.0661 | .0278 | .7294 | .8629 | 50 | .8174 |
| .7563 | 20 | .6862 | .8365 | .9435 | .9747 | 1.0599 | .0253 | .7274 | .8618 | 40 | .8145 |
| .7592 | 30 | .6884 | .8378 | .9490 | .9772 | 1.0538 | .0228 | .7254 | .8606 | 30 | .8116 |
| .7621 | 40 | .6905 | .8391 | .9545 | .9798 | 1.0477 | .0202 | .7234 | .8594 | 20 | .8087 |
| .7650 | 50 | .6926 | .8405 | .9601 | .9823 | 1.0416 | .0177 | .7214 | .8582 | 10 | .8058 |
| .7679 | **44° 00′** | .6947 | .8418 | .9657 | .9848 | 1.0355 | .0152 | .7193 | .8569 | **46° 00′** | .8029 |
| .7709 | 10 | .6967 | .8431 | .9713 | .9874 | 1.0295 | .0126 | .7173 | .8557 | 50 | .7999 |
| .7738 | 20 | .6988 | .8444 | .9770 | .9899 | 1.0235 | .0101 | .7153 | .8545 | 40 | .7970 |
| .7767 | 30 | .7009 | .8457 | .9827 | .9924 | 1.0176 | .0076 | .7133 | .8532 | 30 | .7941 |
| .7796 | 40 | .7030 | .8469 | .9884 | .9949 | 1.0117 | .0051 | .7112 | .8520 | 20 | .7912 |
| .7825 | 50 | .7050 | .8482 | .9942 | .9975 | 1.0058 | .0025 | .7092 | .8507 | 10 | .7883 |
| .7854 | **45° 00′** | .7071 | .8495 | 1.0000 | .0000 | 1.0000 | .0000 | .7071 | .8495 | **45° 00′** | .7854 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIANS |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

# ANSWERS

[Answers which might lessen the value of the Exercise are not given.]

**Pages 9–10.** **5.** $2\frac{1}{4}$ miles. **16.** 173.9 ft.

**Pages 12–13.** **3.** 22. **4.** $\frac{1}{2}(bc + ca + ab)$.

**Pages 16–17.** **4.** $\frac{1}{2} r_1 r_2 \sin (\phi_2 - \phi_1)$.
**5.** $\frac{1}{2}[r_2 r_3 \sin (\phi_3 - \phi_2) + r_3 r_1 \sin (\phi_1 - \phi_3) + r_1 r_2 \sin (\phi_2 - \phi_1)]$.

**Page 21.** **16.** They intersect at $[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)]$.
**19.** $[\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3)]$.

**Pages 40–41.** **6.** 640/39. **9.** $(b_1 m_2 - b_2 m_1)^2/2\, m_1 m_2 (m_1 - m_2)$.
**10.** $(3, \frac{4}{3})$.

**Pages 46–47.** **2.** (*a*) $r \sin \phi = \pm 5$; (*b*) $r \cos \phi = \pm 4$;
(*c*) $r \cos (\phi - \frac{5}{6}\pi) = \pm 12$.
**3.** $\phi = 0$, $r \sin \phi = 9$, $\phi = \frac{1}{2}\pi$, $r \cos \phi = 6$. **14.** 8464/85.
**19.** $(-5, -10)$. **21.** $x = 1$ (by inspection), $4x - 3y + 16 = 0$.

**Page 49.** **4.** $h^2 - ab \gtreqless 0$.

**Page 50.** **1.** $\tan^{-1} \dfrac{2\sqrt{h^2 - ab}}{a + b}$; $a = -b$, $h^2 = ab$.
**4.** $[m_1(b_2 - b) - m_2(b_1 - b)]^2/2\, m_1 m_2 (m_2 - m_1)$.
**6.** $r(2 \cos \phi - 3 \sin \phi) + 12 = 0$.
**10.** 1 hr. 10 m.; 176 miles from Detroit.

**Pages 54–55.** **6.** $x^2 + y^2 - 96x - 54y + 2408 = 0$; 31.8 ft. or 66.3 ft.
**8.** $x^2 + y^2 - 16x + 8y + 60 = 0$. **9.** A circle except for $\kappa = \pm 1$.
**10.** $x^2 + y^2 + 4\dfrac{1 + k^2}{1 - k^2}x + 4 = 0$.

**Page 56.** **2.** (*a*) $r^2 - 20r \sin \phi + 75 = 0$;
(*b*) $r^2 - 12r \cos (\phi - \frac{1}{4}\pi) + 18 = 0$; (*c*) $r + 8 \sin \phi = 0$.

**Page 58.** **3.** $(-6, -1)$, $(29/106, 42/53)$.
**7.** $8x - 4y - 11 \pm 15\sqrt{2} = 0$.

**Page 60.** **3.** $(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$.
**7.** $(-r^2A/C, -r^2B/C)$. **8.** $(2, 1)$.

**Page 62.** **6.** $(x - 79/38)^2 + (y - 55/38)^2 = (65/38)^2$.
**8.** $x^2 + y^2 + 4x - 2y - 15 = 0$.

**Page 67.** **1.** (*c*) Polar lies at infinity.

**Pages 69–70.** **3.** Let $L$, $M$ be the intersections of the circle with $CP_1$, then $d^2 - r^2 = LP_1 \cdot MP_1$.
**4.** $x = y$; $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$.
**6.** (*c*) $2x^2 + 2y^2 + 22x + 6y + 15 = 0$, $2x^2 + 2y^2 - 10x - 10y - 25 = 0$.
**9.** $bx^2 + by^2 + a^2mx - a^2y = 0$.
**12.** If the vertices of the square are $(0, 0)$, $(a, 0)$, $(0, a)$, $(a, a)$ and $k^2$ is the constant, the locus is $2x^2 + 2y^2 - 2ax - 2ay + 2a^2 - k^2 = 0$; $k > a$; $\frac{1}{2}a\sqrt{6}$.
**13.** If the vertices of the triangle are $(a, 0)$, $(-a, 0)$, $(0, a\sqrt{3})$ and $k^2$ is the constant, the locus is $3x^2 + 3y^2 - 2\sqrt{3}\,ay + 3a^2 - 2k^2 = 0$.

**Pages 74–75.** **10.** (*a*) $2y = 3x^2 + 5x$; (*b*) $12y = -5x^2 + 29x - 18$.
**11.** $300y = -x^2 + 230x$; 44.1 ft. above the ground; 230 ft. from the starting point.

**Page 81.** **6.** East, East 33° 41′ North, East 53° 8′ North, East 18° 26′ South.
**10.** $100/(\pi + 4)$.

**Pages 84–85.** **10.** 0, 8° 8′. **11.** 7° 29′.
**15.** When the side of the square is 3 in.
**17.** (*a*) $6y = x^3 + 6x^2 - 19x$; (*b*) $7y = 2x^3 - x^2 - 29x + 35$.

**Page 92.** **10.** $-1.88$, 1.53, .347.

**Pages 97–98.** **2.** (*a*) $(4, \frac{1}{3}\pi)$, $(4, \frac{5}{3}\pi)$; (*b*) $(a, \frac{1}{2}\pi)$, $(a, \frac{3}{2}\pi)$; (*c*) $(4, 0)$; (*d*) $(4a, \frac{1}{3}\pi)$, $(4a, \frac{5}{3}\pi)$.
**7.** (*a*) $y^2 - 4x + 4 = 0$; (*b*) $14y^2 - 45x + 52y + 60 = 0$.
**8.** (*b*) $x^2 - 10x - 3y + 21 = 0$; (*c*) $x^2 + 2x + y - 1 = 0$.
**9.** The equation of a parabola contains an $xy$ term when its axis is oblique to a coordinate axis.

**Pages 106–108.** **8.** (*a*) $y = 0$; (*b*) $2x + 2y - 9 = 0$, $2x - y - 18 = 0$; (*c*) $2x + 2y - 9 = 0$, $8x + 16y - 27 = 0$, $24x - 16y - 153 = 0$. (*d*) $8x - 16y - 27 = 0$.
**14.** $y = kx$. **15.** Directrix; $y^2 = a(x - 3a)$. **21.** $\frac{4a}{m^2}(1 + m^2)$.

**26.** $x^2 - 80x - 2400y = 0$; $0, -\frac{1}{2}, -\frac{2}{3}, -\frac{1}{2}, 0, \frac{5}{6}, 2$.
**29.** $x^2 = 360(y - 20)$.

**Page 115.** **2.** $(3\pi - 4)/6\pi$. **3.** $\frac{8}{3}a^2\frac{(1+m^2)^{\frac{3}{2}}}{m^3}$.
**6.** (*a*) 64/3; (*b*) 625/12; (*c*) 1/12. **7.** 123.84 ft$^3$. **8.** 1794⅓ tons.
**9.** 199.4 ft$^2$.

**Page 118.** **9.** $8x^2 - 2xy + 8y^2 - 63 = 0$.

**Page 122.** **10.** $3x^2 - y^2 = 3a^2$. **11.** $b$. **14.** $2xy = 1$.

**Pages 128–129.** **2.** $\frac{a^2}{x}X + \frac{b^2}{y}Y = c^2$. **13.** 54.5 ft., 42.2 ft. **17.** $b^2/a^2$.
**20.** An ellipse or hyperbola according as one circle lies within or without the other circle.

**Pages 138–139.** **7.** (*a*) $A^2a^2 - B^2b^2 = C^2$; (*b*) $a^2\cos^2\beta - b^2\sin^2\beta = p^2$.
**19.** $b^2$. **21.** $a^2 + b^2$; $a^2 - b^2$. **22.** $4ab$. **23.** $\sin^{-1}(ab/a'b')$.
**25.** (*a*) $x^2 + y^2 = a^2 + b^2$; (*b*) $x^2 + y^2 = a^2 - b^2$.

**Page 144.** **3.** (*a*) $(1, -1)$, $(1 \pm \sqrt{2}, -1)$, $x = 1 \pm \frac{3}{2}\sqrt{2}$;
(*b*) $(\frac{1}{2}, 0)$, $(\frac{5}{2}, 0)$, $(-\frac{3}{2}, 0)$, $x = 0$, $x = 1$.
**4.** $2b^2/a$. **7.** (*a*) $a^2y^2 = b^2x(a - x)$; (*b*) $b^2x^2 = a^2y(b - y)$.
**8.** Two straight lines.

**Page 151.** **2.** (*a*); Vertices (5, 3), (8, 3); semi-axes 3/2, $\sqrt{2}$.
(*b*) Vertices (4, 8/3), (8, 8); semi-axes 10/3, $5\sqrt{3}/3$.
(*c*) Vertices (17/5, 7/5), (1, 3); semi-axes $\sqrt{65}/5$, $\sqrt{13}/2$.
**3.** $3x + 2y - 2 = 0$; 21/13, $-$37/26, $10/\sqrt{13}$.

**Page 153.** **5.** $(a\cos\theta, -a\sin\theta)$, $x^2 + y^2 - 2a(x\cos\theta - y\sin\theta) = 0$.

**Pages 161-162.** **2.** (*a*) $3x - 14y = 0$; (*b*) $y = -3/13$, $x = -14/13$.
**5.** $2x^2 - xy - 15y^2 + x + 19y - 6 = 0$,
$2x^2 - xy - 15y^2 + x + 19y - 28 = 0$.
**6.** $6x^2 + xy - 2y^2 - 9x + 8y - 46 = 0$,
$6x^2 + xy - 2y^2 - 9x + 8y + 34 = 0$.
**11.** (*a*) $x^2/4 + y^2 = 1$; (*b*) $x^2/4 - y^2/2 = 1$; (*c*) $3x^2 + y^2 + 6 = 0$;
(*d*) $x^2/16 + y^2/4 = 1$; (*e*) $(3 + \sqrt{17})x^2 + (3 - \sqrt{17})y^2 = 4$;
(*f*) $(2 + \sqrt{2})x^2 + (2 - \sqrt{2})y^2 = 1$.
**15.** $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.
**19.** Equilateral hyperbola.

**Page 168.** **2.** (*a*) Simple point; (*b*) node; (*c*) cusp; (*d*) cusp. **4.** (*a*) None; (*b*) node at $(b, 0)$; (c) isolated point at $(a, 0)$; (*d*) cusp at $(a, 0)$.

**Pages 174–175.** **4.** $r = a(\sec\phi \pm \tan\phi)$ or $(x-a)y^2 + x^2(x+a) = 0$.
**10.** $x^2y^2 = a^2(x^2+y^2)$. **11.** Cissoid $(a-x)y^2 = x^3$.
**12.** $y(x^2+y^2) = a(x^2-y^2)$. **13.** $r = a \operatorname{ctn}\phi$.

**Page 195.** **6.** $\dfrac{l+l'}{\sqrt{2(1+ll'+mm'+nn')}}$, etc.
**13.** $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$, $\frac{1}{3}(z_1+z_2+z_3)$.

**Page 199.** **6.** $\cos^{-1}(7/3\sqrt{29})$.

**Page 203.** **2.** $\frac{1}{2}\sqrt{465}$. **3.** $\frac{5}{2}\sqrt{269}$.
**6.** $(3962, 47^\circ\,43', 276^\circ\,16')$, $(320, -2914, 2666)$, $2931$.
**7.** $\frac{1}{2}r_1r_2\sqrt{1-[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)]^2}$.
**8.** $\sqrt{r_1^2 + r_2^2 - 2r_1r_2[\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) + \cos\theta_1\cos\theta_2]}$.
**10.** $-1, 10, 7$.

**Page 208.** **3.** $39x - 10y + 7z - 89 = 0$.
**5.** $97/28, -97/49, -97/9$. **7.** $3x - 4y + 2z - 6 = 0$.

**Page 212.** **5.** $4x + 8y + z = 81$, $4x + 8y + z = 90$.

**Page 215.** **2.** (*a*) $56/3$; (*b*) $0$; (*c*) $19/3$.

**Page 218.** **12.** $3x - 2y = 1$. **13.** $6x + 11y + 9z = 58$.
**16.** 70° 31′. **17.** $\cos^{-1}(2h^2 + 3a^2)/(4h^2 + 3a^2)$.

**Pages 226–228.** **3.** 69° 29′. **19.** (*a*) $\sqrt{63/19}$; (*b*) $\sqrt{194/33}$.
**21.** $x - 2y + z + 8 = 0$.
**24.** $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$.

**Pages 236–237.** **4.** $(1, 0, -3)$, $(-9/11, 20/11, 27/11)$.
**7.** $x^2 - 3y^2 - 3z^2 = 0$. **13.** $25(x^2+y^2+z^2) = 16^2$, $25z = 64$.

**Pages 240–241.** **4.** $(4, -5, -3)$. **5.** $(4, 6, 2)$.
**6.** $5x + 2y - z = 25$, $2x - 3y + z + 25 = 0$.

**Page 246.** **4.** $(x^2+y^2+z^2-a^2-b^2)^2 - 4b^2(a^2-y^2) = 0$.
**6.** (*a*) $16a^2(x^2+z^2) = y^4$; (*b*) $16a^2[(x-a)^2+z^2] = (4a^2-y^2)^2$.
**7.** $y^2(x^2+z^2) = a^4$.

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# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .0000 | **0° 00′** | .0000 | —— | .0000 | —— | —— | —— | 1.0000 | .0000 | **90° 00′** | 1.5708 |
| .0029 | 10 | .0029 | .4637 | .0029 | .4637 | 343.77 | .5363 | 1.0000 | .0000 | 50 | 1.5679 |
| .0058 | 20 | .0058 | .7648 | .0058 | .7648 | 171.89 | .2352 | 1.0000 | .0000 | 40 | 1.5650 |
| .0087 | 30 | .0087 | .9408 | .0087 | .9409 | 114.59 | .0591 | 1.0000 | .0000 | 30 | 1.5621 |
| .0116 | 40 | .0116 | .0658 | .0116 | .0658 | 85.940 | .9342 | .9999 | .0000 | 20 | 1.5592 |
| .0145 | 50 | .0145 | .1627 | .0145 | .1627 | 68.750 | .8373 | .9999 | .0000 | 10 | 1.5563 |
| .0175 | **1° 00′** | .0175 | .2419 | .0175 | .2419 | 57.290 | .7581 | .9998 | .9999 | **89° 00′** | 1.5533 |
| .0204 | 10 | .0204 | .3088 | .0204 | .3089 | 49.104 | .6911 | .9998 | .9999 | 50 | 1.5504 |
| .0233 | 20 | .0233 | .3668 | .0233 | .3669 | 42.964 | .6331 | .9997 | .9999 | 40 | 1.5475 |
| .0262 | 30 | .0262 | .4179 | .0262 | .4181 | 38.188 | .5819 | .9997 | .9999 | 30 | 1.5446 |
| .0291 | 40 | .0291 | .4637 | .0291 | .4638 | 34.368 | .5362 | .9996 | .9998 | 20 | 1.5417 |
| .0320 | 50 | .0320 | .5050 | .0320 | .5053 | 31.242 | .4947 | .9995 | .9998 | 10 | 1.5388 |
| .0349 | **2° 00′** | .0349 | .5428 | .0349 | .5431 | 28.636 | .4569 | .9994 | .9997 | **88° 00′** | 1.5359 |
| .0378 | 10 | .0378 | .5776 | .0378 | .5779 | 26.432 | .4221 | .9993 | .9997 | 50 | 1.5330 |
| .0407 | 20 | .0407 | .6097 | .0407 | .6101 | 24.542 | .3899 | .9992 | .9996 | 40 | 1.5301 |
| .0436 | 30 | .0436 | .6397 | .0437 | .6401 | 22.904 | .3599 | .9990 | .9996 | 30 | 1.5272 |
| .0465 | 40 | .0465 | .6677 | .0466 | .6682 | 21.470 | .3318 | .9989 | .9995 | 20 | 1.5243 |
| .0495 | 50 | .0494 | .6940 | .0495 | .6945 | 20.206 | .3055 | .9988 | .9995 | 10 | 1.5213 |
| .0524 | **3° 00′** | .0523 | .7188 | .0524 | .7194 | 19.081 | .2806 | .9986 | .9994 | **87° 00′** | 1.5184 |
| .0553 | 10 | .0552 | .7423 | .0553 | .7429 | 18.075 | .2571 | .9985 | .9993 | 50 | 1.5155 |
| .0582 | 20 | .0581 | .7645 | .0582 | .7652 | 17.169 | .2348 | .9983 | .9993 | 40 | 1.5126 |
| .0611 | 30 | .0610 | .7857 | .0612 | .7865 | 16.350 | .2135 | .9981 | .9992 | 30 | 1.5097 |
| .0640 | 40 | .0640 | .8059 | .0641 | .8067 | 15.605 | .1933 | .9980 | .9991 | 20 | 1.5068 |
| .0669 | 50 | .0669 | .8251 | .0670 | .8261 | 14.924 | .1739 | .9978 | .9990 | 10 | 1.5039 |
| .0698 | **4° 00′** | .0698 | .8436 | .0699 | .8446 | 14.301 | .1554 | .9976 | .9989 | **86° 00′** | 1.5010 |
| .0727 | 10 | .0727 | .8613 | .0729 | .8624 | 13.727 | .1376 | .9974 | .9989 | 50 | 1.4981 |
| .0756 | 20 | .0756 | .8783 | .0758 | .8795 | 13.197 | .1205 | .9971 | .9988 | 40 | 1.4952 |
| .0785 | 30 | .0785 | .8946 | .0787 | .8960 | 12.706 | .1040 | .9969 | .9987 | 30 | 1.4923 |
| .0814 | 40 | .0814 | .9104 | .0816 | .9118 | 12.251 | .0882 | .9967 | .9986 | 20 | 1.4893 |
| .0844 | 50 | .0843 | .9256 | .0846 | .9272 | 11.826 | .0728 | .9964 | .9985 | 10 | 1.4864 |
| .0873 | **5° 00′** | .0872 | .9403 | .0875 | .9420 | 11.430 | .0580 | .9962 | .9983 | **85° 00′** | 1.4835 |
| .0902 | 10 | .0901 | .9545 | .0904 | .9563 | 11.059 | .0437 | .9959 | .9982 | 50 | 1.4806 |
| .0931 | 20 | .0929 | .9682 | .0934 | .9701 | 10.712 | .0299 | .9957 | .9981 | 40 | 1.4777 |
| .0960 | 30 | .0958 | .9816 | .0963 | .9836 | 10.385 | .0164 | .9954 | .9980 | 30 | 1.4748 |
| .0989 | 40 | .0987 | .9945 | .0992 | .9966 | 10.078 | .0034 | .9951 | .9979 | 20 | 1.4719 |
| .1018 | 50 | .1016 | .0070 | .1022 | .0093 | 9.7882 | .9907 | .9948 | .9977 | 10 | 1.4690 |
| .1047 | **6° 00′** | .1045 | .0192 | .1051 | .0216 | 9.5144 | .9784 | .9945 | .9976 | **84° 00′** | 1.4661 |
| .1076 | 10 | .1074 | .0311 | .1080 | .0336 | 9.2553 | .9664 | .9942 | .9975 | 50 | 1.4632 |
| .1105 | 20 | .1103 | .0426 | .1110 | .0453 | 9.0098 | .9547 | .9939 | .9973 | 40 | 1.4603 |
| .1134 | 30 | .1132 | .0539 | .1139 | .0567 | 8.7769 | .9433 | .9936 | .9972 | 30 | 1.4573 |
| .1164 | 40 | .1161 | .0648 | .1169 | .0678 | 8.5555 | .9322 | .9932 | .9971 | 20 | 1.4544 |
| .1193 | 50 | .1190 | .0755 | .1198 | .0786 | 8.3450 | .9214 | .9929 | .9969 | 10 | 1.4515 |
| .1222 | **7° 00′** | .1219 | .0859 | .1228 | .0891 | 8.1443 | .9109 | .9925 | .9968 | **83° 00′** | 1.4486 |
| .1251 | 10 | .1248 | .0961 | .1257 | .0995 | 7.9530 | .9005 | .9922 | .9966 | 50 | 1.4457 |
| .1280 | 20 | .1276 | .1060 | .1287 | .1096 | 7.7704 | .8904 | .9918 | .9964 | 40 | 1.4428 |
| .1309 | 30 | .1305 | .1157 | .1317 | .1194 | 7.5958 | .8806 | .9914 | .9963 | 30 | 1.4399 |
| .1338 | 40 | .1334 | .1252 | .1346 | .1291 | 7.4287 | .8709 | .9911 | .9961 | 20 | 1.4370 |
| .1367 | 50 | .1363 | .1345 | .1376 | .1385 | 7.2687 | .8615 | .9907 | .9959 | 10 | 1.4341 |
| .1396 | **8° 00′** | .1392 | .1436 | .1405 | .1478 | 7.1154 | .8522 | .9903 | .9958 | **82° 00′** | 1.4312 |
| .1425 | 10 | .1421 | .1525 | .1435 | .1569 | 6.9682 | .8431 | .9899 | .9956 | 50 | 1.4283 |
| .1454 | 20 | .1449 | .1612 | .1465 | .1658 | 6.8269 | .8342 | .9894 | .9954 | 40 | 1.4254 |
| .1484 | 30 | .1478 | .1697 | .1495 | .1745 | 6.6912 | .8255 | .9890 | .9952 | 30 | 1.4224 |
| .1513 | 40 | .1507 | .1781 | .1524 | .1831 | 6.5606 | .8169 | .9886 | .9950 | 20 | 1.4195 |
| .1542 | 50 | .1536 | .1863 | .1554 | .1915 | 6.4348 | .8085 | .9881 | .9948 | 10 | 1.4166 |
| .1571 | **9° 00′** | .1564 | .1943 | .1584 | .1997 | 6.3138 | .8003 | .9877 | .9946 | **81° 00′** | 1.4137 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIANS |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

## Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .1571 | **9° 00′** | .1564 | .1943 | .1584 | .1997 | 6.3138 | .8003 | .9877 | .9946 | **81° 00′** | 1.4137 |
| .1600 | 10 | .1593 | .2022 | .1614 | .2078 | 6.1970 | .7922 | .9872 | .9944 | 50 | 1.4108 |
| .1629 | 20 | .1622 | .2100 | .1644 | .2158 | 6.0844 | .7842 | .9868 | .9942 | 40 | 1.4079 |
| .1658 | 30 | .1650 | .2176 | .1673 | .2236 | 5.9758 | .7764 | .9863 | .9940 | 30 | 1.4050 |
| .1687 | 40 | .1679 | .2251 | .1703 | .2313 | 5.8708 | .7687 | .9858 | .9938 | 20 | 1.4021 |
| .1716 | 50 | .1708 | .2324 | .1733 | .2389 | 5.7694 | .7611 | .9853 | .9936 | 10 | 1.3992 |
| .1745 | **10° 00′** | .1736 | .2397 | .1763 | .2463 | 5.6713 | .7537 | .9848 | .9934 | **80° 00′** | 1.3963 |
| .1774 | 10 | .1765 | .2468 | .1793 | .2536 | 5.5764 | .7464 | .9843 | .9931 | 50 | 1.3934 |
| .1804 | 20 | .1794 | .2538 | .1823 | .2609 | 5.4845 | .7391 | .9838 | .9929 | 40 | 1.3904 |
| .1833 | 30 | .1822 | .2606 | .1853 | .2680 | 5.3955 | .7320 | .9833 | .9927 | 30 | 1.3875 |
| .1862 | 40 | .1851 | .2674 | .1883 | .2750 | 5.3093 | .7250 | .9827 | .9924 | 20 | 1.3846 |
| .1891 | 50 | .1880 | .2740 | .1914 | .2819 | 5.2257 | .7181 | .9822 | .9922 | 10 | 1.3817 |
| .1920 | **11° 00′** | .1908 | .2806 | .1944 | .2887 | 5.1446 | .7113 | .9816 | .9919 | **79° 00′** | 1.3788 |
| .1949 | 10 | .1937 | .2870 | .1974 | .2953 | 5.0658 | .7047 | .9811 | .9917 | 50 | 1.3759 |
| .1978 | 20 | .1965 | .2934 | .2004 | .3020 | 4.9894 | .6980 | .9805 | .9914 | 40 | 1.3730 |
| .2007 | 30 | .1994 | .2997 | .2035 | .3085 | 4.9152 | .6915 | .9799 | .9912 | 30 | 1.3701 |
| .2036 | 40 | .2022 | .3058 | .2065 | .3149 | 4.8430 | .6851 | .9793 | .9909 | 20 | 1.3672 |
| .2065 | 50 | .2051 | .3119 | .2095 | .3212 | 4.7729 | .6788 | .9787 | .9907 | 10 | 1.3643 |
| .2094 | **12° 00′** | .2079 | .3179 | .2126 | .3275 | 4.7046 | .6725 | .9781 | .9904 | **78° 00′** | 1.3614 |
| .2123 | 10 | .2108 | .3238 | .2156 | .3336 | 4.6382 | .6664 | .9775 | .9901 | 50 | 1.3584 |
| .2153 | 20 | .2136 | .3296 | .2186 | .3397 | 4.5736 | .6603 | .9769 | .9899 | 40 | 1.3555 |
| .2182 | 30 | .2164 | .3353 | .2217 | .3458 | 4.5107 | .6542 | .9763 | .9896 | 30 | 1.3526 |
| .2211 | 40 | .2193 | .3410 | .2247 | .3517 | 4.4494 | .6483 | .9757 | .9893 | 20 | 1.3497 |
| .2240 | 50 | .2221 | .3466 | .2278 | .3576 | 4.3897 | .6424 | .9750 | .9890 | 10 | 1.3468 |
| .2269 | **13° 00′** | .2250 | .3521 | .2309 | .3634 | 4.3315 | .6366 | .9744 | .9887 | **77° 00′** | 1.3439 |
| .2298 | 10 | .2278 | .3575 | .2339 | .3691 | 4.2747 | .6309 | .9737 | .9884 | 50 | 1.3410 |
| .2327 | 20 | .2306 | .3629 | .2370 | .3748 | 4.2193 | .6252 | .9730 | .9881 | 40 | 1.3381 |
| .2356 | 30 | .2334 | .3682 | .2401 | .3804 | 4.1653 | .6196 | .9724 | .9878 | 30 | 1.3352 |
| .2385 | 40 | .2363 | .3734 | .2432 | .3859 | 4.1126 | .6141 | .9717 | .9875 | 20 | 1.3323 |
| .2414 | 50 | .2391 | .3786 | .2462 | .3914 | 4.0611 | .6086 | .9710 | .9872 | 10 | 1.3294 |
| .2443 | **14° 00′** | .2419 | .3837 | .2493 | .3968 | 4.0108 | .6032 | .9703 | .9869 | **76° 00′** | 1.3265 |
| .2473 | 10 | .2447 | .3887 | .2524 | .4021 | 3.9617 | .5979 | .9696 | .9866 | 50 | 1.3235 |
| .2502 | 20 | .2476 | .3937 | .2555 | .4074 | 3.9136 | .5926 | .9689 | .9863 | 40 | 1.3206 |
| .2531 | 30 | .2504 | .3986 | .2586 | .4127 | 3.8667 | .5873 | .9681 | .9859 | 30 | 1.3177 |
| .2560 | 40 | .2532 | .4035 | .2617 | .4178 | 3.8208 | .5822 | .9674 | .9856 | 20 | 1.3148 |
| .2589 | 50 | .2560 | .4083 | .2648 | .4230 | 3.7760 | .5770 | .9667 | .9853 | 10 | 1.3119 |
| .2618 | **15° 00′** | .2588 | .4130 | .2679 | .4281 | 3.7321 | .5719 | .9659 | .9849 | **75° 00′** | 1.3090 |
| .2647 | 10 | .2616 | .4177 | .2711 | .4331 | 3.6891 | .5669 | .9652 | .9846 | 50 | 1.3061 |
| .2676 | 20 | .2644 | .4223 | .2742 | .4381 | 3.6470 | .5619 | .9644 | .9843 | 40 | 1.3032 |
| .2705 | 30 | .2672 | .4269 | .2773 | .4430 | 3.6059 | .5570 | .9636 | .9839 | 30 | 1.3003 |
| .2734 | 40 | .2700 | .4314 | .2805 | .4479 | 3.5656 | .5521 | .9628 | .9836 | 20 | 1.2974 |
| .2763 | 50 | .2728 | .4359 | .2836 | .4527 | 3.5261 | .5473 | .9621 | .9832 | 10 | 1.2945 |
| .2793 | **16° 00′** | .2756 | .4403 | .2867 | .4575 | 3.4874 | .5425 | .9613 | .9828 | **74° 00′** | 1.2915 |
| .2822 | 10 | .2784 | .4447 | .2899 | .4622 | 3.4495 | .5378 | .9605 | .9825 | 50 | 1.2886 |
| .2851 | 20 | .2812 | .4491 | .2931 | .4669 | 3.4124 | .5331 | .9596 | .9821 | 40 | 1.2857 |
| .2880 | 30 | .2840 | .4533 | .2962 | .4716 | 3.3759 | .5284 | .9588 | .9817 | 30 | 1.2828 |
| .2909 | 40 | .2868 | .4576 | .2994 | .4762 | 3.3402 | .5238 | .9580 | .9814 | 20 | 1.2799 |
| .2938 | 50 | .2896 | .4618 | .3026 | .4808 | 3.3052 | .5192 | .9572 | .9810 | 10 | 1.2770 |
| .2967 | **17° 00′** | .2924 | .4659 | .3057 | .4853 | 3.2709 | .5147 | .9563 | .9806 | **73° 00′** | 1.2741 |
| .2996 | 10 | .2952 | .4700 | .3089 | .4898 | 3.2371 | .5102 | .9555 | .9802 | 50 | 1.2712 |
| .3025 | 20 | .2979 | .4741 | .3121 | .4943 | 3.2041 | .5057 | .9546 | .9798 | 40 | 1.2683 |
| .3054 | 30 | .3007 | .4781 | .3153 | .4987 | 3.1716 | .5013 | .9537 | .9794 | 30 | 1.2654 |
| .3083 | 40 | .3035 | .4821 | .3185 | .5031 | 3.1397 | .4969 | .9528 | .9790 | 20 | 1.2625 |
| .3113 | 50 | .3062 | .4861 | .3217 | .5075 | 3.1084 | .4925 | .9520 | .9786 | 10 | 1.2595 |
| .3142 | **18° 00′** | .3090 | .4900 | .3249 | .5118 | 3.0777 | .4882 | .9511 | .9782 | **72° 00′** | 1.2566 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIANS |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| RADIANS | DEGREES | SINE | | TANGENT | | COTANGENT | | COSINE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .3142 | **18° 00′** | .3090 | .4900 | .3249 | .5118 | 3.0777 | .4882 | .9511 | .9782 | **72° 00′** | 1.2566 |
| .3171 | 10 | .3118 | .4939 | .3281 | .5161 | 3.0475 | .4839 | .9502 | .9778 | 50 | 1.2537 |
| .3200 | 20 | .3145 | .4977 | .3314 | .5203 | 3.0178 | .4797 | .9492 | .9774 | 40 | 1.2508 |
| .3229 | 30 | .3173 | .5015 | .3346 | .5245 | 2.9887 | .4755 | .9483 | .9770 | 30 | 1.2479 |
| .3258 | 40 | .3201 | .5052 | .3378 | .5287 | 2.9600 | .4713 | .9474 | .9765 | 20 | 1.2450 |
| .3287 | 50 | .3228 | .5090 | .3411 | .5329 | 2.9319 | .4671 | .9465 | .9761 | 10 | 1.2421 |
| .3316 | **19° 00′** | .3256 | .5126 | .3443 | .5370 | 2.9042 | .4630 | .9455 | .9757 | **71° 00′** | 1.2392 |
| .3345 | 10 | .3283 | .5163 | .3476 | .5411 | 2.8770 | .4589 | .9446 | .9752 | 50 | 1.2363 |
| .3374 | 20 | .3311 | .5199 | .3508 | .5451 | 2.8502 | .4549 | .9436 | .9748 | 40 | 1.2334 |
| .3403 | 30 | .3338 | .5235 | .3541 | .5491 | 2.8239 | .4509 | .9426 | .9743 | 30 | 1.2305 |
| .3432 | 40 | .3365 | .5270 | .3574 | .5531 | 2.7980 | .4469 | .9417 | .9739 | 20 | 1.2275 |
| .3462 | 50 | .3393 | .5306 | .3607 | .5571 | 2.7725 | .4429 | .9407 | .9734 | 10 | 1.2246 |
| .3491 | **20° 00′** | .3420 | .5341 | .3640 | .5611 | 2.7475 | .4389 | .9397 | .9730 | **70° 00′** | 1.2217 |
| .3520 | 10 | .3448 | .5375 | .3673 | .5650 | 2.7228 | .4350 | .9387 | .9725 | 50 | 1.2188 |
| .3549 | 20 | .3475 | .5409 | .3706 | .5689 | 2.6985 | .4311 | .9377 | .9721 | 40 | 1.2159 |
| .3578 | 30 | .3502 | .5443 | .3739 | .5727 | 2.6746 | .4273 | .9367 | .9716 | 30 | 1.2130 |
| .3607 | 40 | .3529 | .5477 | .3772 | .5766 | 2.6511 | .4234 | .9356 | .9711 | 20 | 1.2101 |
| .3636 | 50 | .3557 | .5510 | .3805 | .5804 | 2.6279 | .4196 | .9346 | .9706 | 10 | 1.2072 |
| .3665 | **21° 00′** | .3584 | .5543 | .3839 | .5842 | 2.6051 | .4158 | .9336 | .9702 | **69° 00′** | 1.2043 |
| .3694 | 10 | .3611 | .5576 | .3872 | .5879 | 2.5826 | .4121 | .9325 | .9697 | 50 | 1.2014 |
| .3723 | 20 | .3638 | .5609 | .3906 | .5917 | 2.5605 | .4083 | .9315 | .9692 | 40 | 1.1985 |
| .3752 | 30 | .3665 | .5641 | .3939 | .5954 | 2.5386 | .4046 | .9304 | .9687 | 30 | 1.1956 |
| .3782 | 40 | .3692 | .5673 | .3973 | .5991 | 2.5172 | .4009 | .9293 | .9682 | 20 | 1.1926 |
| .3811 | 50 | .3719 | .5704 | .4006 | .6028 | 2.4960 | .3972 | .9283 | .9677 | 10 | 1.1897 |
| .3840 | **22° 00′** | .3746 | .5736 | .4040 | .6064 | 2.4751 | .3936 | .9272 | .9672 | **68° 00′** | 1.1868 |
| .3869 | 10 | .3773 | .5767 | .4074 | .6100 | 2.4545 | .3900 | .9261 | .9667 | 50 | 1.1839 |
| .3898 | 20 | .3800 | .5798 | .4108 | .6136 | 2.4342 | .3864 | .9250 | .9661 | 40 | 1.1810 |
| .3927 | 30 | .3827 | .5828 | .4142 | .6172 | 2.4142 | .3828 | .9239 | .9656 | 30 | 1.1781 |
| .3956 | 40 | .3854 | .5859 | .4176 | .6208 | 2.3945 | .3792 | .9228 | .9651 | 20 | 1.1752 |
| .3985 | 50 | .3881 | .5889 | .4210 | .6243 | 2.3750 | .3757 | .9216 | .9646 | 10 | 1.1723 |
| .4014 | **23° 00′** | .3907 | .5919 | .4245 | .6279 | 2.3559 | .3721 | .9205 | .9640 | **67° 00′** | 1.1694 |
| .4043 | 10 | .3934 | .5948 | .4279 | .6314 | 2.3369 | .3686 | .9194 | .9635 | 50 | 1.1665 |
| .4072 | 20 | .3961 | .5978 | .4314 | .6348 | 2.3183 | .3652 | .9182 | .9629 | 40 | 1.1636 |
| .4102 | 30 | .3987 | .6007 | .4348 | .6383 | 2.2998 | .3617 | .9171 | .9624 | 30 | 1.1606 |
| .4131 | 40 | .4014 | .6036 | .4383 | .6417 | 2.2817 | .3583 | .9159 | .9618 | 20 | 1.1577 |
| .4160 | 50 | .4041 | .6065 | .4417 | .6452 | 2.2637 | .3548 | .9147 | .9613 | 10 | 1.1548 |
| .4189 | **24° 00′** | .4067 | .6093 | .4452 | .6486 | 2.2460 | .3514 | .9135 | .9607 | **66° 00′** | 1.1519 |
| .4218 | 10 | .4094 | .6121 | .4487 | .6520 | 2.2286 | .3480 | .9124 | .9602 | 50 | 1.1490 |
| .4247 | 20 | .4120 | .6149 | .4522 | .6553 | 2.2113 | .3447 | .9112 | .9596 | 40 | 1.1461 |
| .4276 | 30 | .4147 | .6177 | .4557 | .6587 | 2.1943 | .3413 | .9100 | .9590 | 30 | 1.1432 |
| .4305 | 40 | .4173 | .6205 | .4592 | .6620 | 2.1775 | .3380 | .9088 | .9584 | 20 | 1.1403 |
| .4334 | 50 | .4200 | .6232 | .4628 | .6654 | 2.1609 | .3346 | .9075 | .9579 | 10 | 1.1374 |
| .4363 | **25° 00′** | .4226 | .6259 | .4663 | .6687 | 2.1445 | .3313 | .9063 | .9573 | **65° 00′** | 1.1345 |
| .4392 | 10 | .4253 | .6286 | .4699 | .6720 | 2.1283 | .3280 | .9051 | .9567 | 50 | 1.1316 |
| .4422 | 20 | .4279 | .6313 | .4734 | .6752 | 2.1123 | .3248 | .9038 | .9561 | 40 | 1.1286 |
| .4451 | 30 | .4305 | .6340 | .4770 | .6785 | 2.0965 | .3215 | .9026 | .9555 | 30 | 1.1257 |
| .4480 | 40 | .4331 | .6366 | .4806 | .6817 | 2.0809 | .3183 | .9013 | .9549 | 20 | 1.1228 |
| .4509 | 50 | .4358 | .6392 | .4841 | .6850 | 2.0655 | .3150 | .9001 | .9543 | 10 | 1.1199 |
| .4538 | **26° 00′** | .4384 | .6418 | .4877 | .6882 | 2.0503 | .3118 | .8988 | .9537 | **64° 00′** | 1.1170 |
| .4567 | 10 | .4410 | .6444 | .4913 | .6914 | 2.0353 | .3086 | .8975 | .9530 | 50 | 1.1141 |
| .4596 | 20 | .4436 | .6470 | .4950 | .6946 | 2.0204 | .3054 | .8962 | .9524 | 40 | 1.1112 |
| .4625 | 30 | .4462 | .6495 | .4986 | .6977 | 2.0057 | .3023 | .8949 | .9518 | 30 | 1.1083 |
| .4654 | 40 | .4488 | .6521 | .5022 | .7009 | 1.9912 | .2991 | .8936 | .9512 | 20 | 1.1054 |
| .4683 | 50 | .4514 | .6546 | .5059 | .7040 | 1.9768 | .2960 | .8923 | .9505 | 10 | 1.1025 |
| .4712 | **27° 00′** | .4540 | .6570 | .5095 | .7072 | 1.9626 | .2928 | .8910 | .9499 | **63° 00′** | 1.0996 |
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | DEGREES | RADIANS |
| | | COSINE | | COTANGENT | | TANGENT | | SINE | | | |

# Four Place Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

| Radians | Degrees | Sine | | Tangent | | Cotangent | | Cosine | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | | |
| .4712 | **27° 00′** | .4540 | .6570 | .5095 | .7072 | 1.9626 | .2928 | .8910 | .9499 | **63° 00′** | 1.0996 |
| .4741 | 10 | .4566 | .6595 | .5132 | .7103 | 1.9486 | .2897 | .8897 | .9492 | 50 | 1.0966 |
| .4771 | 20 | .4592 | .6620 | .5169 | .7134 | 1.9347 | .2866 | .8884 | .9486 | 40 | 1.0937 |
| .4800 | 30 | .4617 | .6644 | .5206 | .7165 | 1.9210 | .2835 | .8870 | .9479 | 30 | 1.0908 |
| .4829 | 40 | .4643 | .6668 | .5243 | .7196 | 1.9074 | .2804 | .8857 | .9473 | 20 | 1.0879 |
| .4858 | 50 | .4669 | .6692 | .5280 | .7226 | 1.8940 | .2774 | .8843 | .9466 | 10 | 1.0850 |
| .4887 | **28° 00′** | .4695 | .6716 | .5317 | .7257 | 1.8807 | .2743 | .8829 | .9459 | **62° 00′** | 1.0821 |
| .4916 | 10 | .4720 | .6740 | .5354 | .7287 | 1.8676 | .2713 | .8816 | .9453 | 50 | 1.0792 |
| .4945 | 20 | .4746 | .6763 | .5392 | .7317 | 1.8546 | .2683 | .8802 | .9446 | 40 | 1.0763 |
| .4974 | 30 | .4772 | .6787 | .5430 | .7348 | 1.8418 | .2652 | .8788 | .9439 | 30 | 1.0734 |
| .5003 | 40 | .4797 | .6810 | .5467 | .7378 | 1.8291 | .2622 | .8774 | .9432 | 20 | 1.0705 |
| .5032 | 50 | .4823 | .6833 | .5505 | .7408 | 1.8165 | .2592 | .8760 | .9425 | 10 | 1.0676 |
| .5061 | **29° 00′** | .4848 | .6856 | .5543 | .7438 | 1.8040 | .2562 | .8746 | .9418 | **61° 00′** | 1.0647 |
| .5091 | 10 | .4874 | .6878 | .5581 | .7467 | 1.7917 | .2533 | .8732 | .9411 | 50 | 1.0617 |
| .5120 | 20 | .4899 | .6901 | .5619 | .7497 | 1.7796 | .2503 | .8718 | .9404 | 40 | 1.0588 |
| .5149 | 30 | .4924 | .6923 | .5658 | .7526 | 1.7675 | .2474 | .8704 | .9397 | 30 | 1.0559 |
| .5178 | 40 | .4950 | .6946 | .5696 | .7556 | 1.7556 | .2444 | .8689 | .9390 | 20 | 1.0530 |
| .5207 | 50 | .4975 | .6968 | .5735 | .7585 | 1.7437 | .2415 | .8675 | .9383 | 10 | 1.0501 |
| .5236 | **30° 00′** | .5000 | .6990 | .5774 | .7614 | 1.7321 | .2386 | .8660 | .9375 | **60° 00′** | 1.0472 |
| .5265 | 10 | .5025 | .7012 | .5812 | .7644 | 1.7205 | .2356 | .8646 | .9368 | 50 | 1.0443 |
| .5294 | 20 | .5050 | .7033 | .5851 | .7673 | 1.7090 | .2327 | .8631 | .9361 | 40 | 1.0414 |
| .5323 | 30 | .5075 | .7055 | .5890 | .7701 | 1.6977 | .2299 | .8616 | .9353 | 30 | 1.0385 |
| .5352 | 40 | .5100 | .7076 | .5930 | .7730 | 1.6864 | .2270 | .8601 | .9346 | 20 | 1.0356 |
| .5381 | 50 | .5125 | .7097 | .5969 | .7759 | 1.6753 | .2241 | .8587 | .9338 | 10 | 1.0327 |
| .5411 | **31° 00′** | .5150 | .7118 | .6009 | .7788 | 1.6643 | .2212 | .8572 | .9331 | **59° 00′** | 1.0297 |
| .5440 | 10 | .5175 | .7139 | .6048 | .7816 | 1.6534 | .2184 | .8557 | .9323 | 50 | 1.0268 |
| .5469 | 20 | .5200 | .7160 | .6088 | .7845 | 1.6426 | .2155 | .8542 | .9315 | 40 | 1.0239 |
| .5498 | 30 | .5225 | .7181 | .6128 | .7873 | 1.6319 | .2127 | .8526 | .9308 | 30 | 1.0210 |
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| | | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Value | Log$_{10}$ | Degrees | Radians |
| | | Cosine | | Cotangent | | Tangent | | Sine | | | |

# Analytic Geometry and Principles of Algebra

BY

**ALEXANDER ZIWET**

PROFESSOR OF MATHEMATICS, THE UNIVERSITY OF MICHIGAN

AND **LOUIS ALLEN HOPKINS**

INSTRUCTOR IN MATHEMATICS, THE UNIVERSITY OF MICHIGAN

**Edited by EARLE RAYMOND HEDRICK**

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